Plabic Graph & Positroid.

- Postnikov 「Total positivity, Grassmannians and networks」.

Setting is distinct from the survey of Speyer. but equivalent

(Postnikov 06).

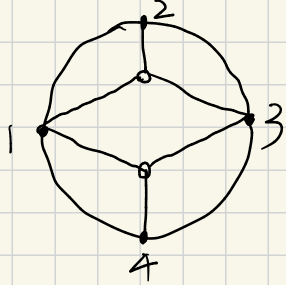
Muller, Speyer 「The twist for positroid varieties」

(Speyer 16).

- Goal: Parametrize $\mathring{\Pi}_{\geq 0}$ in $G(k,n)$.

  Tools: Plabic graphs.

- Def1 : (Plabic graph).



  - Disc.
  - Bipartite planar graph (white & black).
  - Boundary vertices, interior vertices.
    - Boundary vertices: all black, clockwisely labelled by $1, 2, \cdots, n$.
    - Interior vertices: m black, $m+k$ white.

$G(k,n)$

$k \leq n$.

- Def2: A perfect matching M is a subset of $\{\text{edges of } G\}$ s.t.
  - every interior vertex of G lies on exactly one edge of M
  - every boundary vertex of G lies on at most one edge of M.

(So there are exactly k boundary vertices lying on exactly one edge of M.)

- Def3: Given a weight $w(e) \in \mathbb{F}$ for $\forall e \in G$.

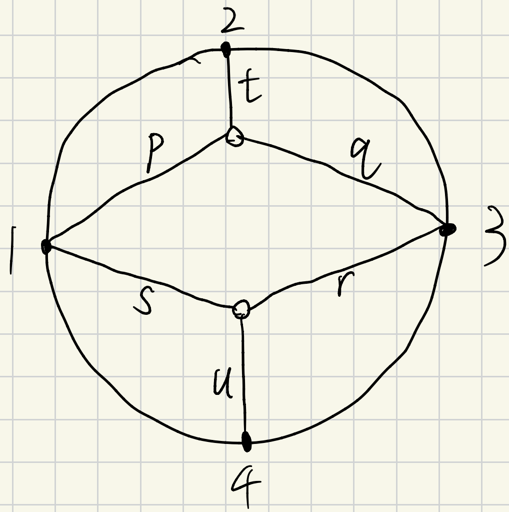
$$w(M) = \prod_{e \in M} w(e)$$

$\forall I \in \binom{[n]}{k}$, define $D_I := \sum_{\partial M = I} w(M)$.

Δ_I (Plücker coordinates of $G(k,n)$).

e.g.:

$G =$



$D_{12} = st \qquad D_{13} = pr + qs \qquad D_{14} = pu$

$D_{23} = tr \qquad D_{24} = tu \qquad D_{34} = qu.$

- Thm 1. (Thomas Lam 「Totally nonnegative Grassmannian and Grassmann polytopes」).

For any values of the weights $w(e)$ in the ground field $\mathbb{F}$, either all of the D_I are zero, or else there is a $L \in G(k,n)$ s.t. $\Delta_I(L) = D_I$ for all $I \in \binom{[n]}{k}$

$$(G, \{w(e)\}_e) \longmapsto (D_I)_{I \in \binom{[n]}{k}}.$$

Either 0 or corresponds to a $L \in G(k,n)$.

Fix G, $\mu : \{w(e)\}_e \longmapsto L \in G(k,n)$

$$\mathbb{F}^{Edge(G)} \dashrightarrow G(k,n).$$

How to obtain a well-defined morphism?

- Assume that G has at least one perfect matching. and consider $(\mathbb{R}_{>0})^{Edge(G)}$ instead of $\mathbb{F}^{Edge(G)}$.
- Thm 2 (Postnikov 06).

Rmk: Vice versa! Thm 3.1 of Speyer16.

The image of $\mu : (\mathbb{R}_{>0})^{Edge(G)} \longrightarrow G(k,n)$

("boundary measurement map). (Fix G !)

is $\mathring{\Pi}_{\geq 0} \subseteq G(k,n)$. for some $\mathring{\Pi} \subseteq G(k,n)$.

However $\mu : \mathbb{R}_{>0}^{\mathrm{Edge}(G)} \twoheadrightarrow \mathring{\Pi}_{\geq 0} \subseteq G(k,n)$.

(Fix G). $\{w(e)\} \longmapsto (\Delta_I)_{I \in \binom{[n]}{k}}$

$\updownarrow$ Plücker coordinates

$L \in G(k,n)$.

is rarely bijective.

- Def 4: Two weight functions $w, w' \in \mathbb{G}_m^{\mathrm{Edge}(G)}$ are <u>gauge equivalent</u> if there is a function

$$t : \{\text{Vertices of } G\} \longrightarrow \mathbb{G}_m$$

s.t. $t|_{\partial G} \equiv 1$

s.t. $\forall$ edge e of G with end vertices (v, w), we have $w'(e) = w(e) \cdot t(v) \cdot t(w)$.

$$T := \mathbb{G}_m^{\mathrm{Edge}(G)} \Big/ (\text{gauge equivalence}).$$

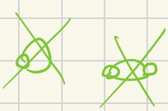
$$T_{>0} := \mathbb{R}_{>0}^{\mathrm{Edge}(G)} \Big/ (\text{gauge equivalence}).$$

- Then, μ descends to

$$\mu : T_{>0} \twoheadrightarrow \mathring{\Pi}_{\geq 0}$$

T is a torus depending on G.

$T \cong \mathbb{G}_m^{\square}$ (G is simple



$\square = (\#\text{ of faces of } G) - 1$

- We hope that $\mu: T_{>0} \twoheadrightarrow \mathring{\Pi}_{\geq 0}$ is bijective giving a diffeomorphism.

"Reduced plabic graphs".

- Def/Thm 5: If $\boxed{G} \rightsquigarrow \boxed{\mathring{\Pi}} \subseteq G(k,n)$, TFAE.

$G \rightsquigarrow (\mu : \mathbb{R}_{>0}^{Edge(G)} \to G(k,n).)$

$Im(\mu) = \mathring{\Pi}_{\geq 0} \subseteq G(k,n)$

$\mathring{\Pi}$

(1) $\underbrace{\dim T}_{\text{purely combinatorial}} = \dim \mathring{\Pi}$

(2) $\mu : T_{>0} \longrightarrow \mathring{\Pi}_{\geq 0}$ is bijective

(3): $\mu : T_{>0} \longrightarrow \mathring{\Pi}_{\geq 0}$ is a diffeomorphism.

In this case, we call G reduced.

- Four crucial things:

Solved

(1): Given any plabic graph $G' \rightsquigarrow \mathring{\Pi}_{\geq 0}$

How to convert G' into a reduced plabic graph G, perserving $Im(\mu) = \mathring{\Pi}_{\geq 0}$.

(2) How to determine whether G is reduced only using G itself combinatorially.

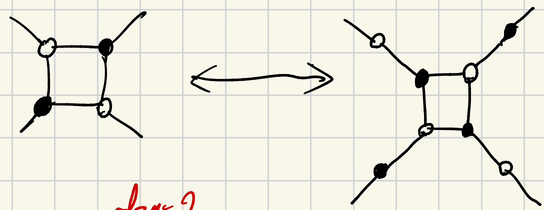
(3): For a reduced $G \rightsquigarrow \mathring{\Pi} \rightsquigarrow f \in Bound(k,n)$. find f directly from G.

(4) (Open). (Postnikov 06) For a general G, do (3).

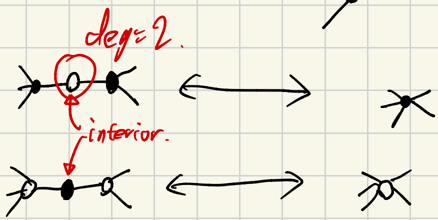
• For (1), We define two classes of operations

Move — Reduction.

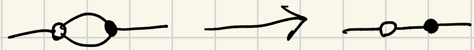
Move : — M_1 :



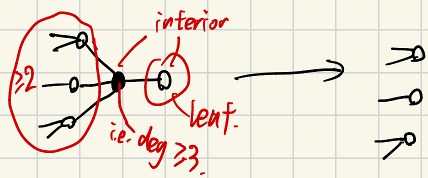
M_2 :

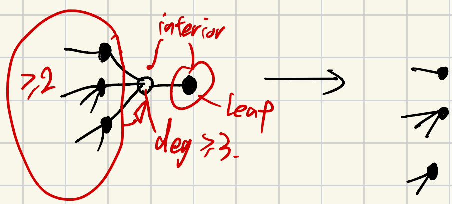


Reduction : — R_1 :

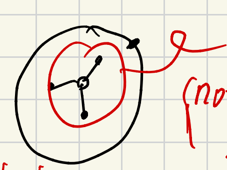


R_2 :

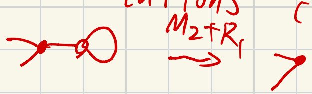




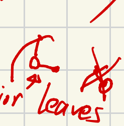
R_3 : Delete isolated interior connected components.



(Rmk: These relations can help us



& delete interior leaves


)

∘ Thm 3: (Postnikov 06 / Speyer 16).

Let $G \rightsquigarrow \mathring{\pi}$, $G' \rightsquigarrow \mathring{\pi}$, but G is reduced.

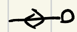
Then $\exists$ a series of M_i's and R_i's converting G' into G.

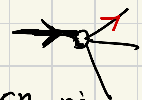
In particular, if G' is also reduced, then $\{M_i\}$ is enough.

For (2): We use zig-zag path given below:

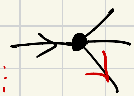
- Def 6: A zig-zag path: directed path. (May be a closed circle).

  From any edge, choose a direction and extend along both sides whenever possible.

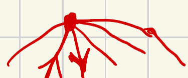
  - Leaf: Turn 180° 
  - Not Leaf:
    - White: turn left as sharply as possible. ($\cancel{180}^\circ$)

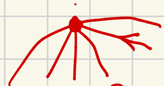
    - Black: turn right as sharply as possible. ($\cancel{180}^\circ$)

When to stop?



DO Not stop



STOP.

STOP.

∘

∘ Thm 4:

Let G be a plabic graph. Then G is reduced iff all the following hold:

(1) G has no interior leaves.

(2) No zig-zag circles.

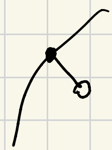
(Rmk (1) & (2) $\Rightarrow$ G has no isolated interior connected components)

(3) For each boundary vertex $i \in [n]$, one of the following happens:

Rmk: Any Reduced graph is simple & has no interior leaves.

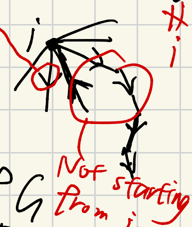
(I) $\deg i = 0$

(II) $\deg i = 1$ & i is incident to a leaf



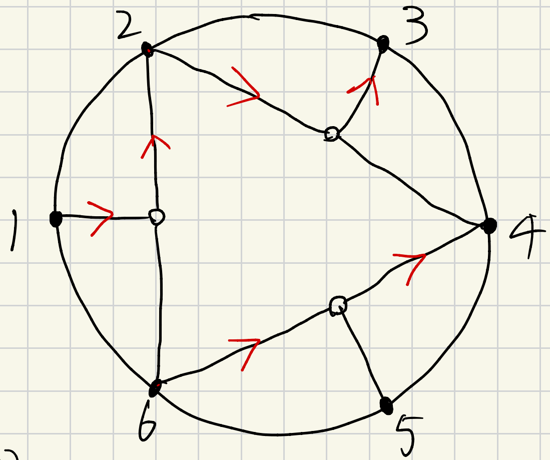
(III) $\exists$ a zig-zag path starting from i and ending at another boundary vertex j $\neq i$

(4) Each zig-zag path does not pass through any edge twice.



(5) If α, β are two distinct edges of G lying on two distinct zig-zag paths P_1, P_2. Then α, β occur in opposite orders in P_1 and P_2.

e.g.



$k=3$

$n=6.$

$f(1)\equiv 3 \mod 6$

$1\le f(1)\le 1+6$

$f(1)=3.$

$f(2)\equiv 6 \mod 6$

$2\le f(2)\le 2+6$

$f(2)=6.$

- For (3).

Thm 5:

If G is reduced, then G corresponds to Π_f where $f\in \text{Bound}(k,n)$ is given by:

$$\begin{cases} \circ\ f:\mathbb{Z}\to\mathbb{Z},\ \text{periodic } n. \\ \circ\ i\le f(i)\le i+n,\ \forall i\in\mathbb{Z}. \\ \circ\ \frac{1}{n}\sum_{i=1}^{n}(f(i)-i)=k. \end{cases}$$

$\forall\, i\in[n] \rightsquigarrow f(i).$

Case 1: $i \xrightarrow{\text{zig-zag starting from } i \text{ \& ending at } j\neq i} $ ← Boundary

$(j \mod n)$

[Why unique? Because $f(i)\in[i,i+n]$, $f(i)\equiv j \mod n$. The choice of $f(i)$ is unique.]

Case 2: $i \xrightarrow{\deg i\in\{0,1\}} \begin{cases} i, & \deg i=0 \\ i+n, & \deg i=1 \end{cases}$

e.g. Look at the example above.

$$f(i)=\begin{cases} i+2, & \text{for } i \text{ odd} \\ i+4, & \text{for } i \text{ even.} \end{cases}$$

- Recall that: $\mu : T_{>0} \longrightarrow \mathring{\Pi}_{\geq 0}$.

What about $T = \mathbb{G}_m^{Edeg(G)} / (\text{gauge equivalence})$

For G reduced, it's OK! (algebraic)

- Thm 6: (Speyer 16).

Let G be a reduced plabic graph corresponding positroid variety Π_f.

Then: $\mu : T \longrightarrow \mathring{\Pi}_f$ is an open inclusion.

$$\begin{array}{ccc} \mathbb{G}_m^{Edeg(G)} / (\text{gauge}). & & \\ \mathbb{G}_m^{Edge(G)} & \longrightarrow & \mathring{\Pi}_f \\ \Psi & & \\ w & \longmapsto & (D_I)_{I \in \binom{[n]}{k}} \end{array}$$

Plücker coordinates $\rightsquigarrow L \in G(k,n)$.

Rmk: $Im(\mu)$ is open in $G(k,n)$. but is not determined by Plücker coordinates. [i.e. $\{L : \Delta_I(L) \neq 0 \text{ for } I \in S\}$.)

- Thm 7: (Speyer 16).

  $\exists$ albegraic automorphism τ of $\mathring{\Pi}_f$ (called "twist").

  s.t. a chain isom

$$T \xrightarrow{\mu} \mu(T) \xrightarrow{\tau} \Omega \xrightarrow{\sigma} \mathbb{G}_m^{\#\{\text{Faces of } G\}-1}$$

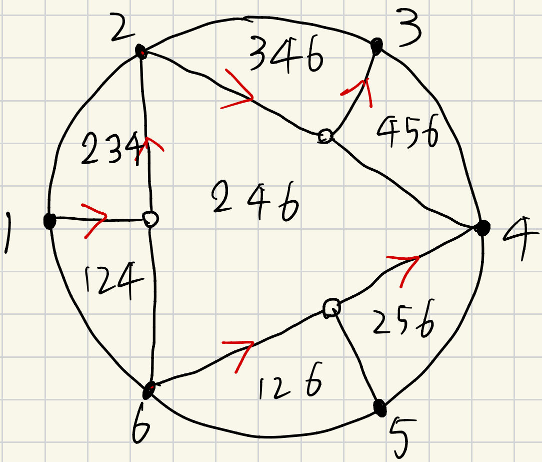
b/c: Projective

Ω = of the form $\{L : \Delta_I(L) \neq 0 \text{ for } I \in S\}$

I

$|S| = \#\{\text{faces of } G\}$.

- Describle S.



$I_F \in \binom{[n]}{k}$ for each face F

Δ_{I_F}.

For each face, we assign $I \in \binom{[n]}{k}$ to it.
given by: $\forall i \in [n]$, we have $i \in I$ iff zig-zag $f^{-1}(i) \to i$ is on the right.

- Recall: Grassmann Necklace.

$$(I_1, \cdots, I_n)$$
$$\rotatebox{90}{$\subseteq$}\quad \rotatebox{90}{$\subseteq$}$$
$$\binom{[n]}{k}.$$

Let $(I_1, \cdots, I_n)$ be the Grassmann Necklace indexing $\mathring{\Pi}_f$.

Given $[M] \in \mathring{\Pi}_f$ $\quad M \in Mat_{n\times k}(\mathbb{C})$

$$M = \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_n \end{pmatrix}.$$

$\forall\, i \in [n]$, $\quad \exists!\ \vec{w}_i \in \mathbb{C}^n$ s.t.

$$\vec{w}_i \cdot \vec{v}_j = \begin{cases} 1, & i = j \\ 0, & j \in I_i \setminus \{i\} \end{cases}$$

Let $\tau([M]) = \left[\begin{bmatrix} \vec{w}_1 \\ \vdots \\ \vec{w}_n \end{bmatrix}\right] \in \mathring{\Pi}_f$

$$\tau : \mathring{\Pi}_f \xrightarrow{\simeq} \mathring{\Pi}_f.$$