* [Chen-Fan-Xiong-Y] arXiv 2511.00980

BPD Fragments - Equivariant Geometry of Clans

* [Wyser-Yong] Polynomials for $GL_p \times GL_q$-orbits closures in ~~the~~ Flag variety

1. Introduction
2. Clan — as index of $\underset{\sim}{K}$-orbits ($K = GL_p \times GL_q$)
   - Poly. repr.
   - BPD fragment
3. Main results.

1. Introduction

* (Spherical subgroup).

$\underset{\text{Borel subgroup}}{B} \leq G$ reductive group, $H \leq G$ is called spherical if $|H \backslash G/B| = \#\{H\text{-orbit of } gB \in \underset{\text{Flag variety}}{G/B}\} < \infty$

* Eg: $B \leq G$ is spherical (Bruhat decomposition)

* Symmetric Group.

$\theta: G \to G$ automorphism, $\theta^2 = id$

A Spherical subgroup K is symmetric if $K = G^\theta = \{g \mid \theta(g) = g\}$

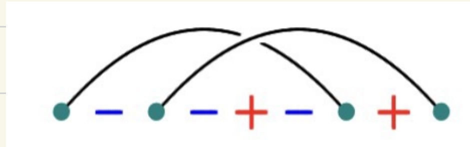
call. (K, G) symmetric pair

Example: $G = GL_*$

Type	Involution	(K, G).	Index set of $K\backslash G/B$
AI	$g \mapsto (g^t)^{-1}$	(O_n, GL_n)	$\{w \in S_n ; w^2 = 1\}$ – involution perm
AII	$g \mapsto \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} (g^t)^{-1} \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$	(Sp_{2n}, GL_{2n})	$\{w \in S_{2n} : w^2 = 1, w(i) \neq i\}$ – fix point free involution perm
AIII	$g \mapsto \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix} \cdot g \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$	$(GL_p \times GL_q, GL_{p+q})$	$\{(p, q)\text{–clan}\}$.

Today.

2. Clans



$\swarrow$ $(4,5)$-clan.

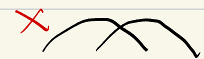
* (Clans) A (p,q)-clan is a partial matching of $p+q$ nodes with unmatched node labeled by $+$ or $-$ such that $\#\{+\} - \#\{-\} = p-q$

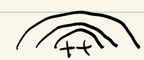
$\Updownarrow$

$\#\{+\} + \#\{\text{matching}\} = p$

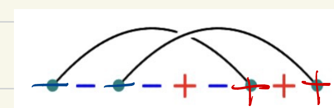
$\#\{-\} + \#\{\text{matching}\} = q$.

* matchless : $+ - + - - \cdots$

non-crossing : 

rainbow :  or 

* (W_r & U_r).



(a) left endpoint $\mapsto +$, right endpoint $\mapsto -$ $\Rightarrow$ $V_r^{-1} = (\text{indices}+)(\text{indices}-)$

$V_r = 1\ 5\ 2\ 6\ 3\ 7\ 8\ 4\ 9$.

$U_r = 1\ 2\ 3\ 4\ 6\ 5\ 7\ 8\ 9$.

left endpoint $\mapsto -$, right endpoint $\mapsto +$ $\Rightarrow$ $U_r^{-1} = (\text{indices}-)(\text{indices}+)$

Rmk: $V_r = id$ iff $r \overset{(a)}{\Rightarrow} r' = \overbrace{++\cdots+}^{p}\overbrace{--\cdots-}^{q}$

2.1. Clan idex K-orbits $(K = GL_p \times GL_q,\ n = p+q)$.

[Matsuki] $\quad GL_n/B = \bigsqcup_{\gamma} \underbrace{Y_\gamma^\circ}_{K\text{-orbit}}$

$\ast$ Def: (clan $\gamma \longrightarrow V_\bullet^\gamma$).

For a (p,q)-clan γ, construct $V_\bullet^\gamma = (V_1^\gamma \subset \cdots \subset V_i^\gamma \subset \cdots V_n^\gamma) \in G/B$, $V_i^\gamma = V_{i-1}^\gamma \oplus \mathbb{C}\, v_i^\gamma$, where

- $v_i^\gamma = e_{v_\gamma(i)} + e_{v_\gamma(j)}$ if (i,j) is a matching and i is a left endpoint
- $v_i^\gamma = e_{v_\gamma(i)}$ if i is $\pm$ or right endpoint.

Let $\dot\gamma = (v_1^\gamma, v_2^\gamma, \cdots, v_n^\gamma)$. $V_\bullet^\gamma = \dot\gamma B/B$. $Y_\gamma^\circ = K\cdot\dot\gamma B/B$.

$Y_\gamma = \overline{Y_\gamma^\circ}$ denote the Zariski closure of Y_γ°

$$\begin{array}{cccccccccc} v_\gamma = & 1 & 5 & 2 & 6 & 3 & 7 & 8 & 4 & 9. \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ & e_1 & e_5 & e_2 & e_6 & e_3 & e_7 & e_8 & e_4 & e_9. \\ & + & & + & & & & & & \\ & e_8 & & e_9 & & & & & & \end{array}$$

Rmk:

γ non crossing $\Rightarrow Y_\gamma = X_{v_\gamma}^{w_0 u_\gamma}$.

2.2. Polynomial representative.

Recall: $T \subset B^-$. $[X_w]_T \in H_T^*(Fl_n)$

$\Big\downarrow$ Poly. repr.

$$\mathfrak{S}_w(x;y) = \begin{cases} \prod_{i+j\le n}(x_i - y_j) & \text{if } w = w_0 \\ \partial_i \mathfrak{S}_{ws_i}(x;y) & \text{if } w < ws_i \end{cases}$$

$$\partial_u \mathfrak{S}_w(x;y) = \begin{cases} \mathfrak{S}_{wu^{-1}}(x;y) & \text{if } \ell(wu^{-1}) = \ell(w) - \ell(u) \\ 0 & \text{otherwise.} \end{cases}$$

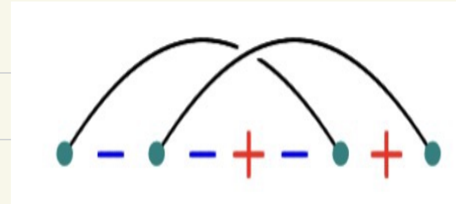
For $[Y_r]_T$: [Wyser & Yong]. $T \subset K$ $[Y_r]_T \in H_T^*(Fl_n)$

$\Big\downarrow$ Poly. repr.

?

* Weak order on clans.

1° length of Clan:



length = 11.

$$\ell(\gamma) = \sum_{(i,j)\text{-matching}} (j-i) - \#\,\text{crossing in } \gamma$$

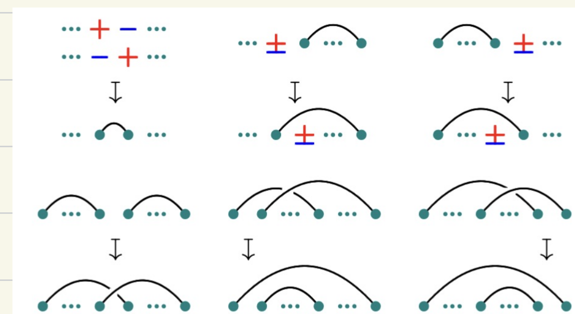
Yamamoto
Orbits in flag variety and images of the moment map for classic group.

$$= \dim Y_\gamma - \frac{p(p-1)}{2} - \frac{q(q-1)}{2}.$$

2° 0-Hecke action on clan.

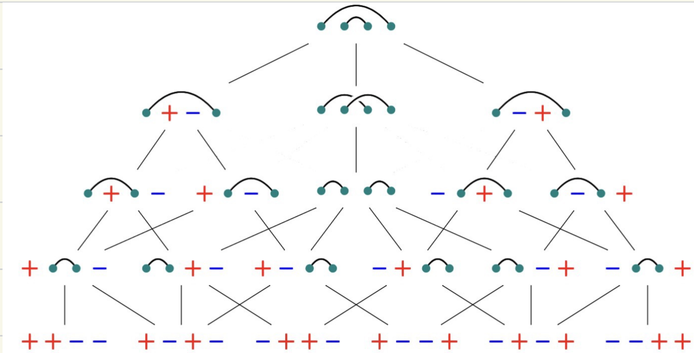
obtained by exchanging i-th node and $(i+1)$-th node of γ.

$$\forall i \in [1, p+q-1],\quad s_i * \gamma = \begin{cases} \gamma' & \text{if } \ell(\gamma') > \ell(\gamma). \\ \gamma & \text{otherwise.} \end{cases}$$



3° cover: $s_i * \gamma > \gamma$ if $\ell(s_i * \gamma) > \ell(\gamma)$

Example: (2,2)-clan. (weak order).



[Wyser] Strong order.

The Bruhat order on clans.

$*$ Poly. repr. of $[Y_r]_T$:

Given a (p,q)-clan, with $p+q=n$, the poly. repr. $\Upsilon_r(x;y)$ of $[Y_r]_T$ is.

$$\Upsilon_r(x;y) = \mathfrak{S}_{u_r}(x;\overleftarrow{y})\cdot \mathfrak{S}_{v_r}(x;y) \quad \text{if } r \text{ is matchless} \qquad \overleftarrow{y} = (y_n \cdots y_1)$$

$$\Upsilon_{s_i * r}(x;y) = \underbrace{\partial_i}_{\text{acts on } x \text{ only}} \Upsilon_r(x;y) \quad \text{if } s_i * r > r.$$

$$\partial_u \Upsilon_r(x;y) = \begin{cases} \Upsilon_{u*r}(x;y) & \text{if } \ell(u*r) - \ell(r) = \ell(u) \\ 0 & \text{otherwise} \end{cases} \qquad u*r = s_{i_1} * \cdots * s_{i_{\ell(u)}} * r.$$

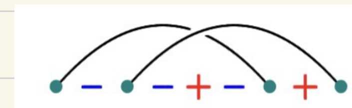
2.3. Clans and BPD fragment.

✱ Def (Clan to BPD fragment) Give a (p,q)-clan.

1° $\lambda(\gamma) \subset p \times q$ rectangle, γ gives a path from NE corner to SW corner

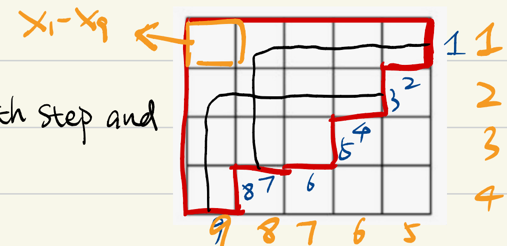
node i $+$ or left ends $\to$ i-th step is a south step

node i $-$ or right ends $\to$ i-th step is a west step

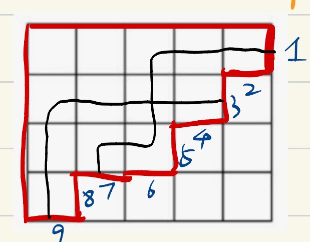


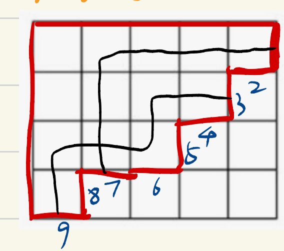
2° Rothe BPD fragment.

(i,j)-step $\longrightarrow$ pipe with only one ┌ connect i-th step and j-th step



3° BPD(γ): droop from Rothe BPD fragment.





* (Clan polynomial).

$$S_r(x) = S_r(x_1, \cdots x_n) := \sum_{\pi \in BPD(r)} wt(\pi). \qquad wt(\pi) = \prod_{ij \text{ is } \square} (x_i - x_{n-j+1})$$

$$S_r(x) = (x_1 - x_9)(x_3 - x_7) + (x_1 - x_9)(x_2 - x_9) + (x_1 - x_9)(x_1 - x_8).$$

[MS].

Rmk: $v_r = id$. $S_r(x_1 \cdots x_p, y_q, \cdots y_1)$ is double schubert polynomial for partial permutation.

w_r: (i,j)-matching
$\Updownarrow$
$i \mapsto p+q+1-j$

3. Main result.

localization Formula:

* localization: $-|_w : H_T^*(Fl_n) \to H_T^*(\{V_\bullet^w\}) \cong \mathbb{Q}[y_1, \cdots y_n]$.

· in poly repr: $x_i \mapsto y_{w(i)}$.

eg: $[X_u]_T|_w = \mathfrak{S}_u(x,y)|_w = \mathfrak{S}_u(y_{w(1)} \cdots y_{w(n)}; y_1, \cdots y_n)$.

* Prop. $Y_r(x,y)|_w = 0$, unless $v_r \le w \le w_0 u_r$. $wB/B \notin Y_r$ if $[Y]_T|_w = 0$

By proposition, $[Y_r]_T|_{id} = 0$ unless $v_r = id$

Main result 1.

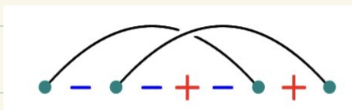
For any (p,q)-clan γ.

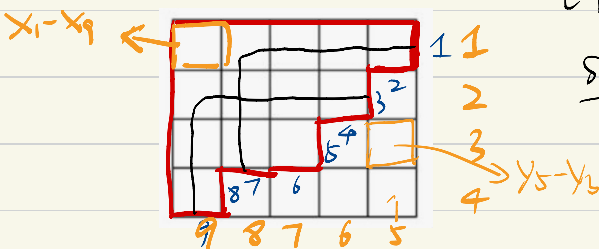
$$[Y_\gamma]|_{v_\gamma} = Y_\gamma(x;y)|_{v_\gamma} = S_\gamma(y)\cdot \prod_{(i,j)\in\overline{\lambda(\gamma)}} (y_{n-j+1}-y_i).$$

$\overline{\lambda(\gamma)}$: complement of $\lambda(\gamma) \subset p\times q$.

$S_\gamma(x)$

Example:





$$[Y_\gamma]_T|_{v_\gamma} = S_\gamma(y)\cdot(y_5-y_2)(y_5-y_3)(y_5-y_4)(y_6-y_3)(y_6-y_4)(y_7-y_4)(y_8-y_4)$$

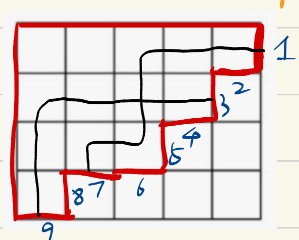
<u>Sketch of proof</u>

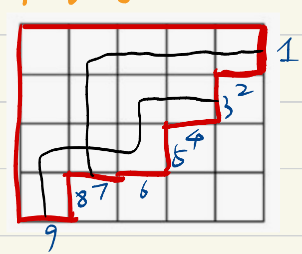
Step 1. $[Y_\gamma]|_{id}$. $H_T^*(G/B) \xrightarrow{-|_{id}} H_T^*(1B/B)$.

$\Downarrow$

For a (p,q)-clan, $v_\gamma = id$

$$[Y_\gamma]_T|_{id} = Y_\gamma(x;y)|_{id} = S_\gamma(y).$$





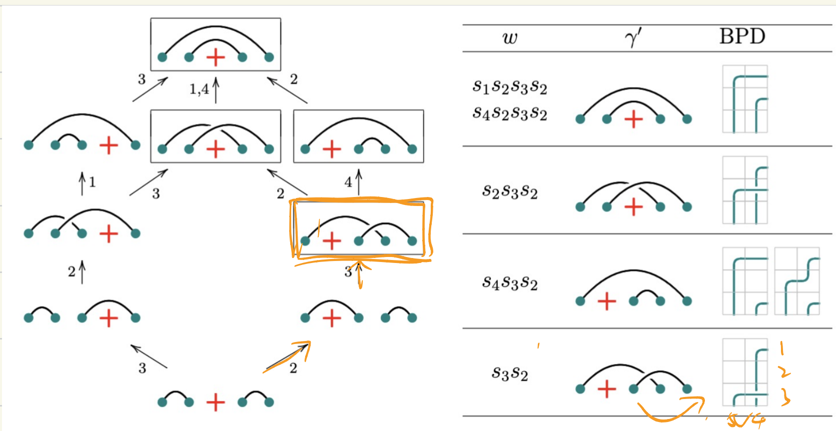
Step 2.

$$Y_\gamma(x;y)|_w = Y_\gamma(x;y)|_{ws_i} + \begin{cases} (y_{w(i)}-y_{w(i+1)})\cdot Y_{s_i\times\gamma}(x;y)|_{ws_i} & \text{if } s_i\times\gamma \to \gamma \\ 0 & \text{otherwise} \end{cases}$$

Main result 2.

ThmB: For any (p,q)-clan γ, $\Upsilon_\gamma(x,y) = \sum_{w\in S_n} c_{\gamma,w}(y)\cdot\mathfrak{S}_w(x,y)$,

$$c_{\gamma,w}(y) = \begin{cases} S_{\gamma'}(y) & \text{if } \gamma' = w*\gamma,\ v_{\gamma'}=id\ \&\ \ell(\gamma')-\ell(\gamma)=\ell(w). \\ 0 & \text{otherwise}. \end{cases}$$



$$\Upsilon_\gamma(x,y) = \mathfrak{S}_{s_1s_2s_3s_2} + \mathfrak{S}_{s_4s_2s_3s_2} + (y_1-y_5)\cdot\mathfrak{S}_{s_2s_3s_2} \quad ((y_1-y_5)+(y_2-y_4))\mathfrak{S}_{s_4s_3s_2} + (y_1-y_5)(y_2-y_5)\mathfrak{S}_{s_3s_2}$$

Corollary of Thm B.

For any non-crossing clan γ and $v \in S_n$, assume that

$$\mathfrak{S}_{u_\gamma}(x;y)\cdot\mathfrak{S}_v(x;y) = \sum d^{w}_{u_\gamma,v}(y)\cdot\mathfrak{S}_w(x;y).$$

$$d^{w_0 v_\gamma}_{u_\gamma,v}(y) = C_{\gamma,w_0 v}(\overleftarrow{y}).$$

Y_γ°

Thm D. For any γ. TFAE.

1) Y_γ smooth at V_γ.

2) γ has exactly one BPD fragment

3) γ avoids the following 5 patterns