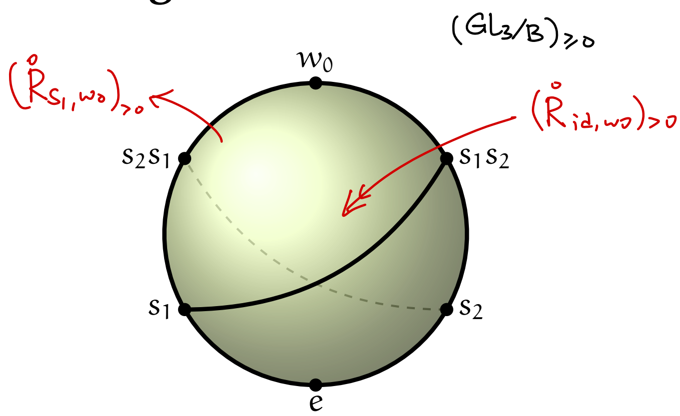
Topology of TTN flag varieties

November 28, 2025

1 Background



Background

Compare: $\mathbb{CP}^1, \mathbb{RP}^1, (\mathbb{RP}^1)_{\geq 0}$

Lusztig: ''remarkable polyhedral subspace''

GKL: $(G/P)_{\geq 0}$ a ball

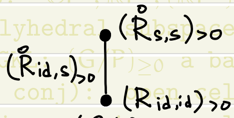
Rietsch (Lusztig conj): open cells

Williams: $(G/P)_{\geq 0}$ regular

$Gl_2/B \cong \mathbb{P}^1$

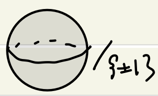
$\mathbb{CP}^1$

$\mathbb{RP}^1$

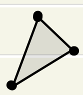


$(G/B)_{\geq 0}$

$Gl_3/P \cong \mathbb{P}^2$



$\mathbb{RP}^2$



$(G/P)_{\geq 0}$

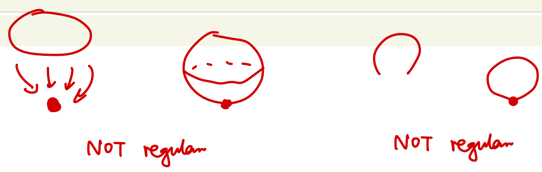
Lusztig "remarkable polyhedral subspace"

GKL $(G/P)_{\geq 0}$ is a ball

Rietsch $(G/P)_{\geq 0}$ can be decomposed into open cells. $(\mathring{R}_{u,w})_{>0}$

Williams $(G/P)_{\geq 0}$ is a regular CW with respect to this cell decomp. (proved by GKL)

Regular CW : closure of any its cells is a closed ball (cell $\leftrightarrow$ open ball)



2 Sketch of the proof

References

[1] Galshin, Karp, Lam. Regularity theorem for totally nonnegative flag varieties.

[2] Bao, He. Product structure and regularity theorem for totally nonnegative varieties.

2.1 Poincaré conjecture

Generalized

e.g. 

Let X be a compact n-dimensional topological manifold with boundary. Then

$$\left.\begin{array}{ll} \partial X \cong \partial D^n = S^{n-1} & \text{(sphere)} \\ X \setminus \partial X \cong D^n \setminus \partial D^n & \text{(open ball)} \end{array}\right\} \Rightarrow X \cong D^n \quad \text{(closed ball)}$$

(known: $(\mathring{R}_{u,w})_{>0} \cong \mathbb{R}^{\ell(w)-\ell(u)}$)

2.2 Topological Input

prove each cell closure is a topological manifold with boundary.

$$\overline{(\mathring{R}_{u,w})_{>0}} = \bigsqcup_{u \leq u' \leq w' \leq w} (\mathring{R}_{u',w'})_{>0}$$

$$\partial \overline{(\mathring{R}_{u,w})_{>0}} = \bigsqcup_{\substack{u \leq u' \leq w' \leq w \\ (u,w) \neq (u',w')}} (\mathring{R}_{u',w'})_{>0}$$

Main step: show $\overline{(\mathring{R}_{u,w})_{>0}}$ is a topological manifold with boundary.

Induction : $\overline{(\mathring{R}_{u,w})_{>0}}$ is regular CW.

2.3 Combinatorial Input

Prove the order is graded, shellable and thin.

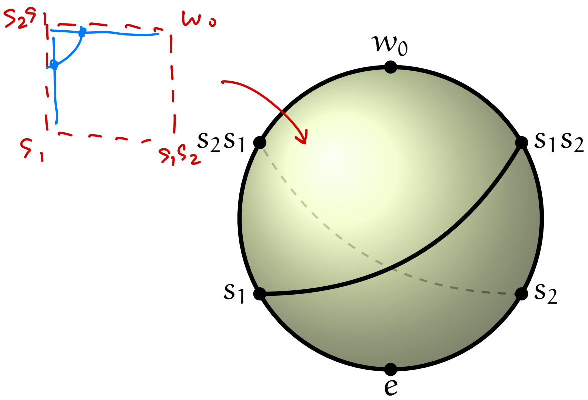
$Q = \{(u,w) : u \leq w\}$ index $(\mathring{R}_{u,w})_{>0}$

$(u,w) \leq (u',w') \Leftrightarrow u \leq u' \leq w' \leq w$ closure relation

Main Step: prove $Q \sqcup \{0,1\}$ is graded, thin, shellable.

($\Rightarrow$ if Q is a regular CW X face poset $\Rightarrow$ X is a sphere)

3 Topological part (sketch)



Step 1. Check $\overline{(\mathring{R}_{u,w})_{>0}}$ is topological manifold locally.

$$\overline{(\mathring{R}_{u,w})_{>0}} \cap v\underbrace{U^- B/B}_{\text{opposite Schubert cell of codim } =0} \qquad (u \le v \le w)$$

$$= \bigsqcup_{u \le u' \le v \le w' \le w} (\mathring{R}_{u',w'})_{>0}$$

GL_n $\quad B = \begin{bmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{bmatrix}$ $\quad U^- = \begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ * & \cdots & 1 \end{bmatrix}$

$U^- B = B^- B$ $\quad B^- = \begin{bmatrix} * & & \\ \vdots & \ddots & \\ * & \cdots & * \end{bmatrix}$

Step 2 "Product structure"

$$(\mathring{R}_{u,w})_{>0} \cong (\mathring{R}_{u,v})_{>0} \times (\mathring{R}_{v,w})_{>0} \qquad \mathring{R}_{w_0w, w_0v} // \ w_0v$$

$$v \longmapsto (v \ , \ v)$$

$$\underbrace{\mathring{R}_{u,w}}_{\ell(w)-\ell(u)} \cap vU^-B/B \cong \underbrace{\mathring{R}_{u,v}}_{\ell(v)-\ell(u)} \times \underbrace{\mathring{R}_{v,w}}_{\ell(w)-\ell(v)}$$

Question reduce to the case $u=v$ or $w=v$.

Step 3 $\quad (\mathring{R}_{u,w})_{>0}$ around u.

$Lk = (\mathring{R}_{u,w})_{>0} \cap$ small sphere around u.

$$\overline{(\mathring{R}_{u,w})_{>0}} \cap uU^-B/B = \underline{\text{Cone}}(\overline{Lk})$$

$$\text{Cone}(X) = X \times [0,1] / X \times \{1\}$$

4 Combinatorics of posets

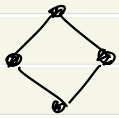
4.1 Properties of posets

Let P be a poset.

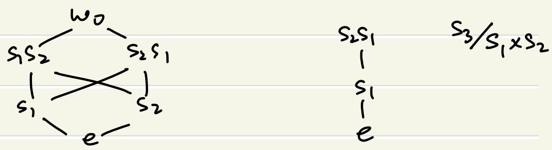
(1) P is graded if any maximal chain has the same length.

($\Rightarrow P = \bigsqcup_{i \in \mathbb{Z}} P_i$)

(2) P is thin if every interval of rank 2 has exactly 4 elements



Example: Bruhat order on W.



Warning: Parabolic Bruhat order is NOT thin in general

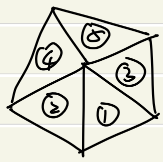
(3) P is shellable if its $\underbrace{\text{order complex}}_{\text{a simplicial complex}}$ is shellable.

$\{\text{chains in } P\} \subseteq 2^P$

A r-dim complex is shellable if its maximal faces can be ordered $F_1, \cdots, F_n$

$$F_k \cap (F_1 \cup \cdots \cup F_{k-1})$$

is a nonempty union of $(r-1)$-dim faces



e.g. polytope shellable

e.g. Bruhat order

(4) P is EL-shellable, if

($\mathcal{E}(P) = \{\text{edges in Hasse diag}\}$)

$\exists$ labeling $\mathcal{E}(P) \longrightarrow$ totally ordered set

▷ $\exists$ a maximal increasing chain in every interval.

▷ any other maximal chain $\le_{lex}$ it.

(EL shellability $\Rightarrow$ shellability).

4.2 Two theorems of Björner

① For a poset Q with $Q \cup \{\hat{0}\}$ graded, thin, shellable, then Q is a CW poset, i.e. Q is a face poset of a regular CW complex.

formal minimum

If $Q\cup\{0\}$ GTS $\Rightarrow$ Q is a CW poset

If $Q\cup\{0,1\}$ GTS $\Rightarrow$ Q is a CW poset for a sphere

Prove of combinatorial part

② If X is regular CW complex with face poset Q

If $Q \cup \{\hat{0}, \hat{1}\}$ is graded, thin & shellable

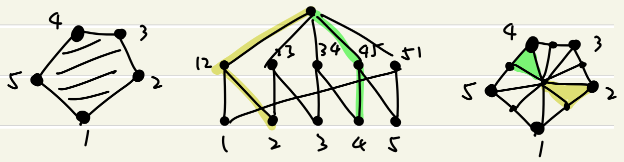
formal maximum

$\Rightarrow$ X is a sphere.

Rmk. regular CW complex $\longrightarrow$ face poset

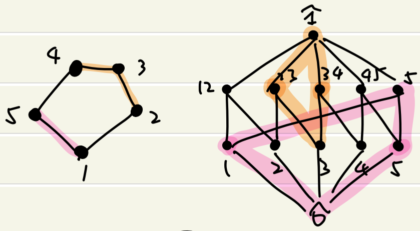
order complex $\longleftarrow$ poset

"$\supseteq$" = barycentric subdivision

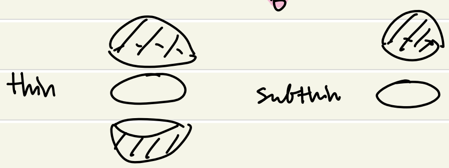


Rmk subthin = not thin & every interval of rank 2 has ≤ 4 elements

② "sphere" $\longrightarrow$ "closed ball"



$Q \cup \{\hat{0}, \hat{1}\}$



$Q = \{ (u,w) : u \le w \}$

$(u,w) \le (u',w') \iff u \le u' \le w' \le w$

$Q \cup \{\hat{0}\}$ is graded & thin & shellable

$(Q \setminus \{(id, w_0)\}) \cup \{\hat{0}, \hat{1}\}$

$Q \hookrightarrow W \times W$ image is saturated

$(u,w) \longmapsto (u, w_0 w)$

Since $W \times W$ is graded & thin & shellable

$\Rightarrow$ Q is graded & thin & shellable

Not hard $Q \cup \{\hat{0}\}$ is graded & thin & shellable

4.3 Parabolic case (sketch)

embed to affine bruhat order.

$(G/P)_{\ge 0} = \bigsqcup_{u \le w \in W^P} (\Pi^{\circ}_{u,w})_{\ge 0}$

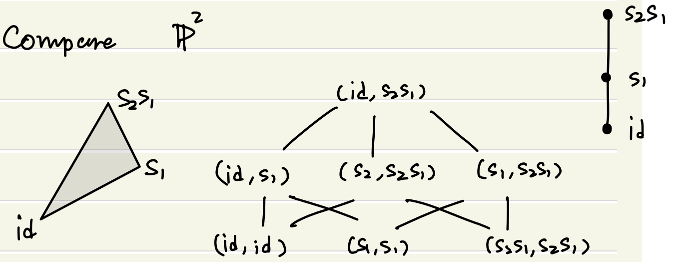
$Q = \{ (u,w) : u \le w \text{ and } w \in W^P \}$

$(u,w) \le (u',w') \iff \exists v \in W_P$ s.t.

$u \le u'v \le w'v \le w$

(interval inclusion up to W_P-shift)

Compare $\mathbb{P}^2$



$Q \longrightarrow W_{ext}$ = extended affine Weyl group

$(u,w) \longrightarrow u t_\lambda w^{-1}$

λ t_λ

▷ $W_{ext} = W \ltimes$ (coweight lattice)

$W \circlearrowright$ coweight lattice

λ is chosen such that

stabilizer of $\lambda = W_P$

▷ $uvt_\lambda(wv)^{-1} = u(vt_\lambda v^{-1})w^{-1}$

$= u t_\lambda w^{-1}$

$W t_\lambda W = W t_\lambda W^P \overset{1:1}{\longleftrightarrow} W \times W^P$

▷ preserving order proven by [He Lam].

Since W_{ext} is graded & thin & shellable

$\Rightarrow$ Q is graded & thin & shellable

Not hard $Q \cup \{\hat{0}\}$ is graded & thin & shellable

Conclusion: $(G/P)_{\ge 0}$ is a regular CW complex.