# Cluster structure on the Grassmannian

## 1. Cluster algebra.

- quiver: an oriented graph without
  (箭(筒)) 1-cycle and 2-cycle.

$$\begin{tikzcd} 1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} & 4 & \\ 1 \arrow[r, shift left] \arrow[r, shift right] & 2 \arrow[u] \arrow[r] & 3 \end{tikzcd}$$

- mutation

$\mu_k$: ① $i \to k \to j$ adding an edge $i \to j$

② invert all arrows incident to $k$.

③ delete 2-cycles.

$$\begin{tikzcd} 3 \arrow[r] & 1 \arrow[d] \\ 4 \arrow[u] & 2 \arrow[l] \end{tikzcd} \xrightarrow{\mu_4} \begin{tikzcd} 3 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 4 \arrow[r] & 2 \arrow[ul] \end{tikzcd} \xrightarrow{\mu_2} \begin{tikzcd} 3 \arrow[dr] & 1 \\ 4 & 2 \arrow[l] \arrow[u] \end{tikzcd}$$

$\mu_k(\mu_k(Q)) = Q$.

- seed $(\vec{x}, Q)$, $\vec{x} = (x_1, \cdots x_n, x_{n+1}, \cdots, x_m)$

  mutatable vertex, forzen vertex

  cluster variable forzen variable (coefficient)

$$\mu_k(\vec{x}, Q) = (\vec{x}', \mu_k(Q))$$

$$x_k' = \frac{\prod_{k \to j} x_j + \prod_{i \to k} x_i}{x_k}$$

$$\underset{x_1}{\circ} \longrightarrow \underset{x_2}{\circ} \quad \xrightarrow{\mu_1} \quad \underset{x_1'}{\circ} \longleftarrow \underset{x_2}{\circ}$$

$$x_1' = \frac{x_2 + 1}{x_1}$$

<u>cluster</u> (⺮), cluster variable. coefficients.

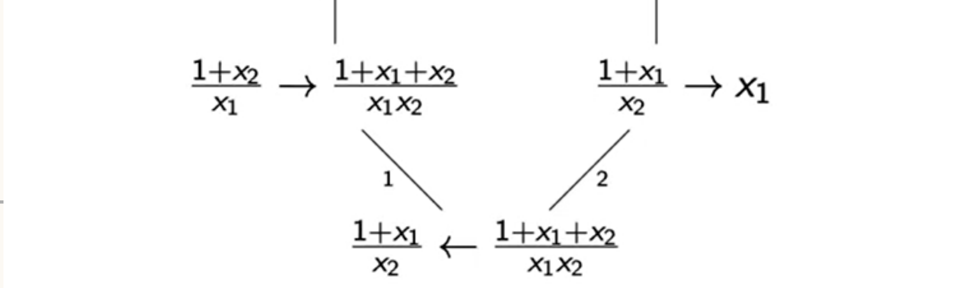
- cluster algebra

$$A_Q = \mathbb{C}[\text{forzen variable}^{\pm}][\text{cluster variable}]$$

all possible clusters

$$\begin{tikzcd}
 & x_1 \to x_2 \arrow[dl, dash, "1"'] \arrow[dr, dash, "\cong"] & \\
\frac{1+x_2}{x_1} \leftarrow x_2 \arrow[d, dash, "2"'] & & x_2 \leftarrow x_1 \arrow[d, dash, "1"] \\
{} & & {}
\end{tikzcd}$$

5 cluster

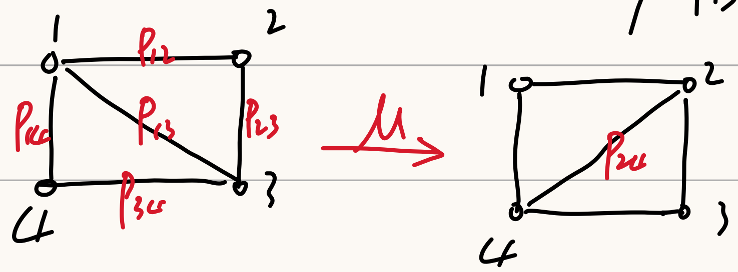



$$\mathcal{A}_Q = \left\langle x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}, \frac{1+x_1}{x_2} \right\rangle \subset \mathbb{Q}(x_1, x_2).$$

2. cluster structure on $Gr(2,n)$

- $\mathbb{C}[Gr(2,4)] = \mathbb{C}[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}] / \text{Plücker}$

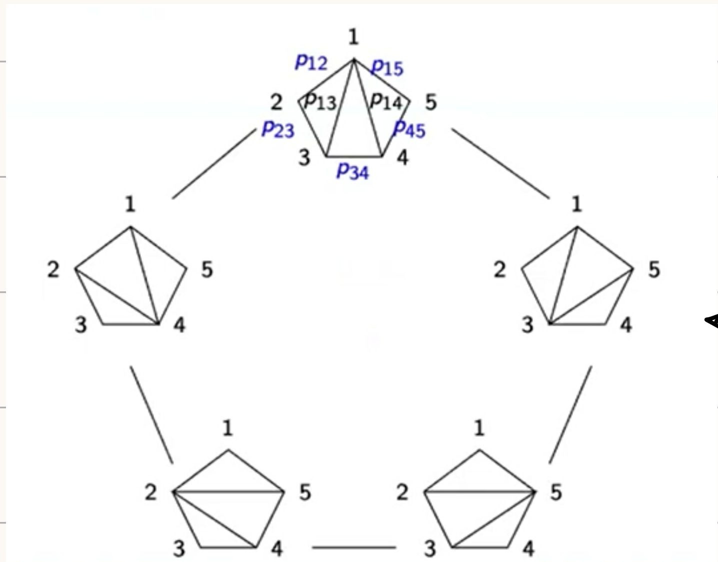
$$= \mathbb{C}[p_{ij}]_{1\le i,j\le 4} / p_{13}\underline{p_{24}} = p_{12}p_{34} + p_{23}p_{14}$$




triangulation

$$= \mathbb{C}\left[p_{12}, p_{23}, p_{34}, p_{14}, p_{13}, \frac{p_{12}p_{34}+p_{23}p_{14}}{p_{13}}\right]$$

$Gr(2,5)$

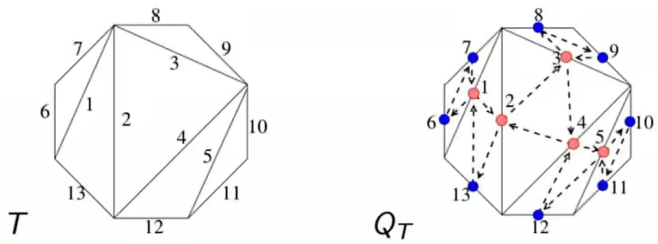



associahera

generalized ...ahedra

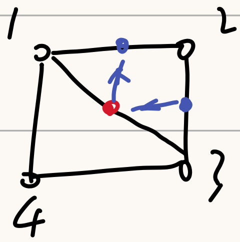
associahedra

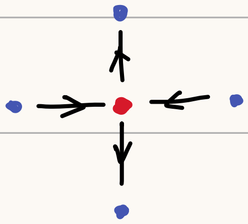
# Theorem [Fomin–Zelevinsky] $Gr(2, n)$

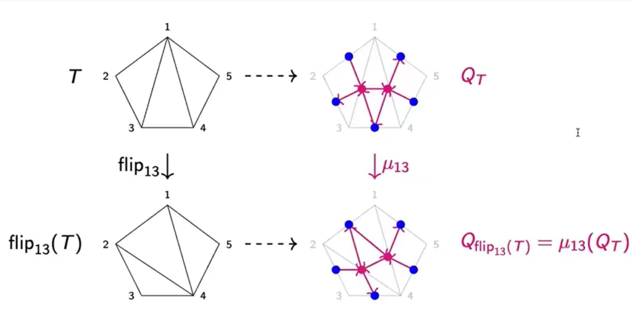


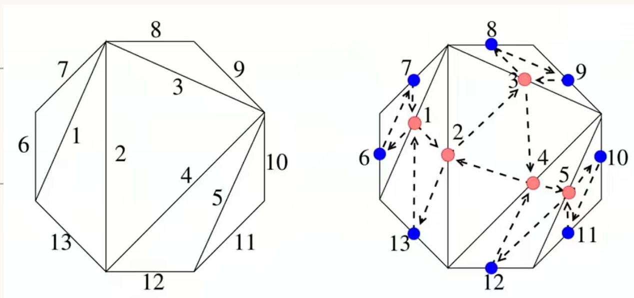

Triangulations are connected by flips, and flips $\leftrightarrow$ mutations. Moreover:

$$\begin{aligned} \text{seeds} &\leftrightarrow \text{the triangulations of the polygon} \\ \text{coefficient variables} &\leftrightarrow \text{the } d \text{ sides of the polygon} \\ \text{cluster variables} &\leftrightarrow \text{the } \frac{d(d-3)}{2} \text{ diagonals of the polygon} \end{aligned}$$

$T$: 

$Q_T$: 



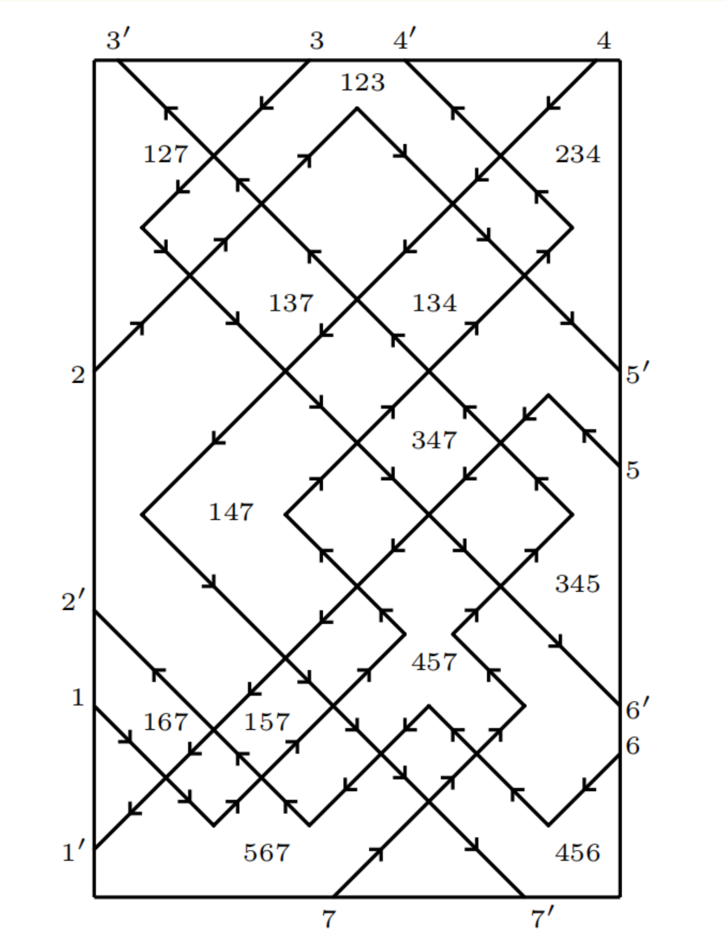



- cluster algebra on $Gr(k, n)$

( J. Scott 2006. PLMS. )

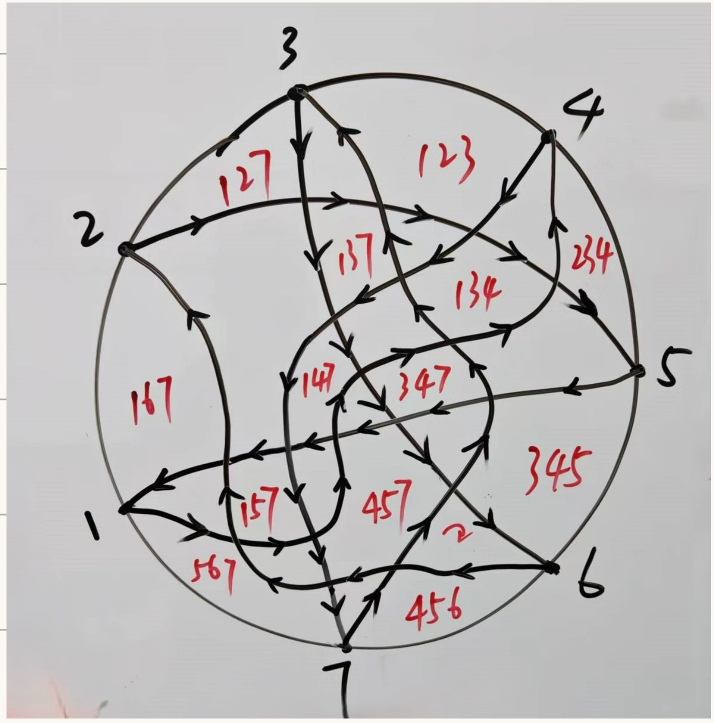
# Postnikov diagram

$$\pi_{k,n} = \begin{matrix} 1 & 2 & 3 & \cdots & n \\ k{+}1 & k{+}2 & \cdots & 1 \cdots & k \end{matrix}$$




$\pi_{3,7}$ - diagram

$(k=3,\ n=7)$



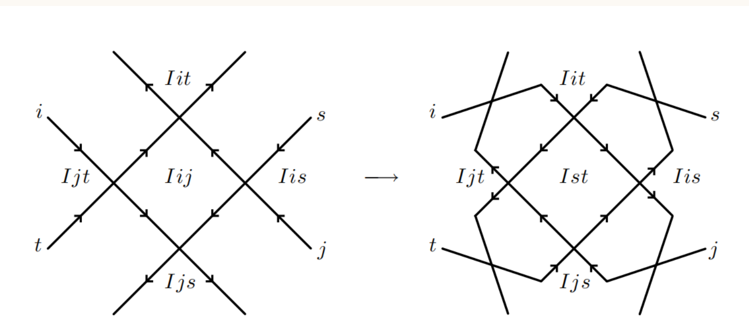
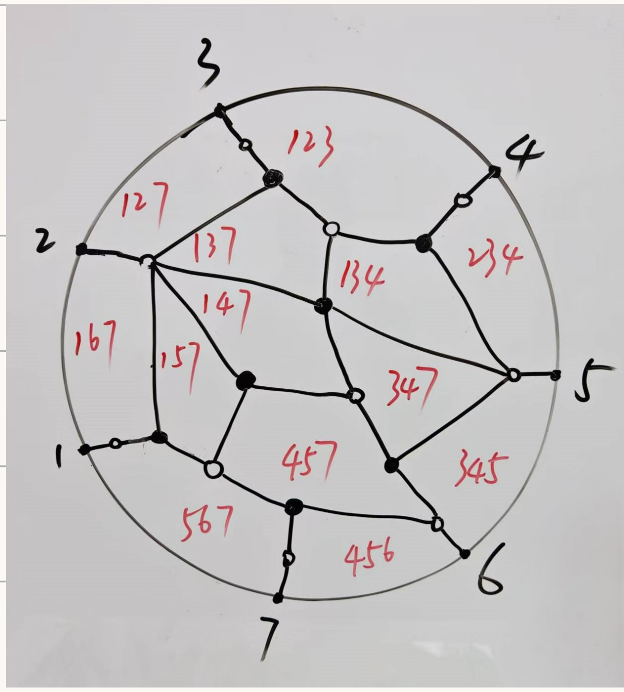



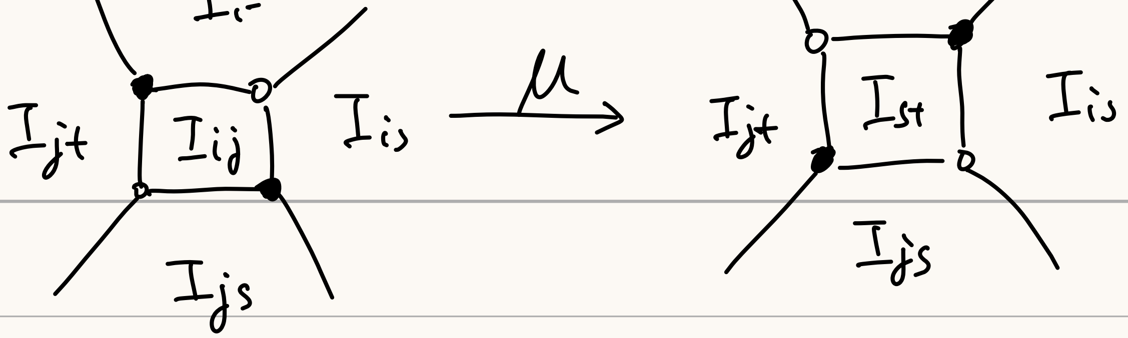

FIGURE 7. *Geometric exchange.*

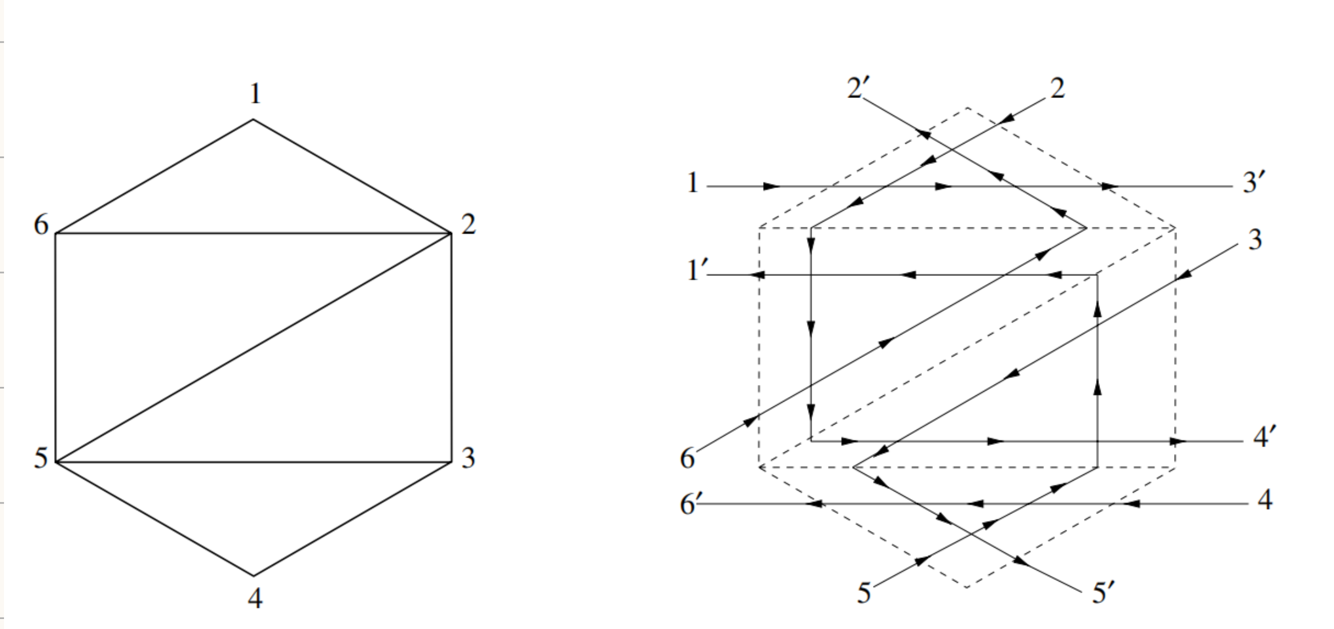


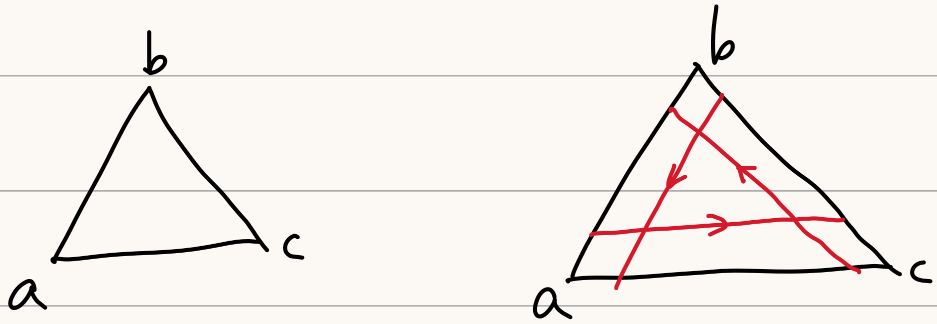

$|I| = k-2,$

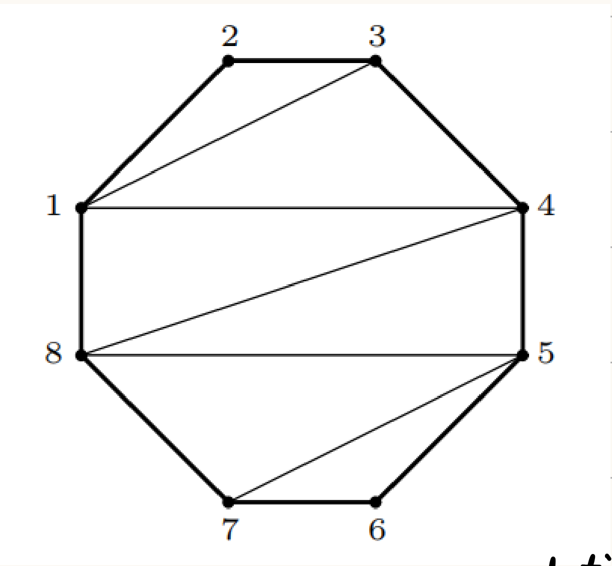
$I \cap \{i,j,s,t\} = \phi$

$I_{it}$

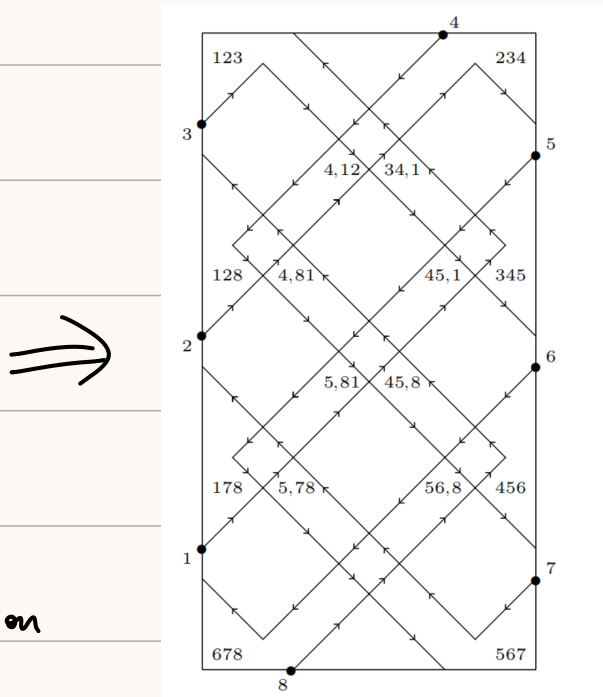


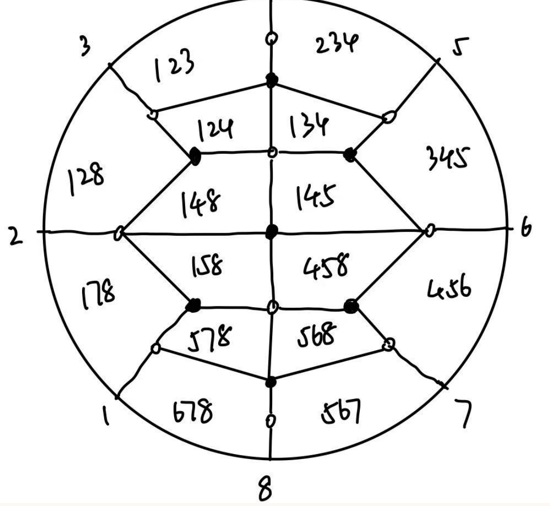



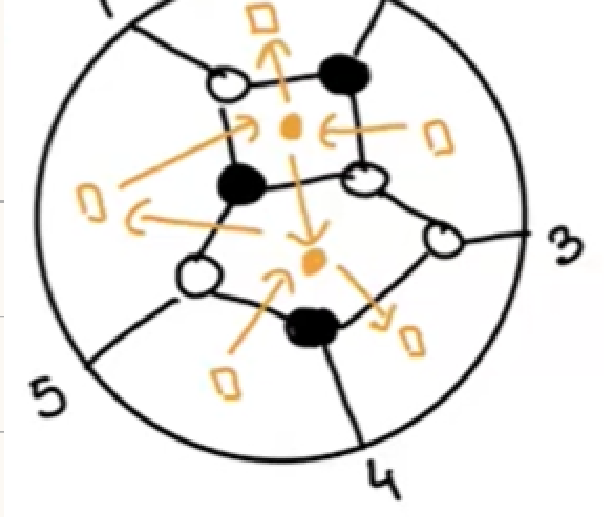




Zig-zag triangulation




Akin

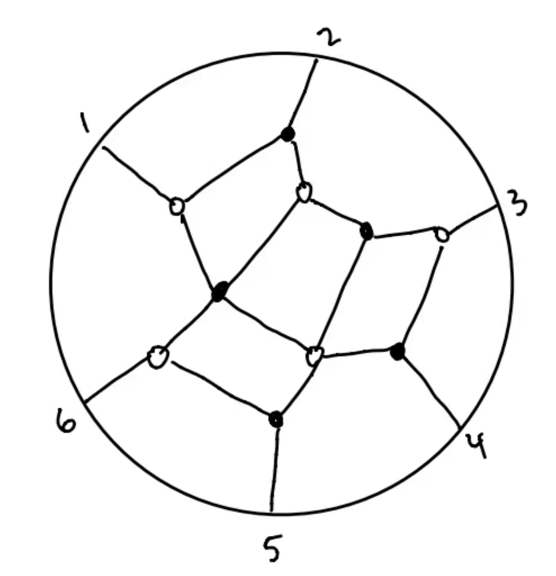



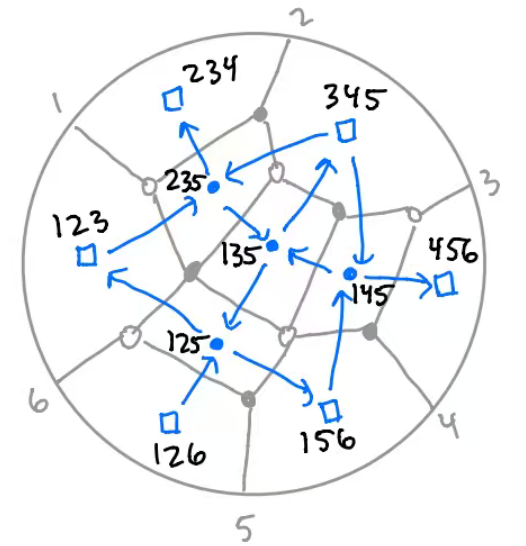




THEOREM 3. *There is an isomorphism*

$$\varphi : \mathcal{A}_{k,n} \longrightarrow \mathbb{C}[\mathbb{G}(k,n)]$$

*of* $\mathbb{C}\big[[K_1],\dots,[K_n]\big]$*- algebras with the property that* $\varphi$ *maps* $[K]$ *to* $\Delta^K$ *for every* $k$*-subset* $K$ *of* $[1\dots n]$.




PROPOSITION 1 (Fomin and Zelevinsky). *Let* $\mathcal{A}$ *be a rank* $N$ *cluster algebra, let* $\mathcal{X}$ *denote the set of all cluster variables, and let* $\mathbf{c}$ *be the set of coefficients. Let* $X$ *be a rational quasi-affine irreducible algebraic variety over* $\mathbb{C}$. *Assume that we are given a family of regular functions on the variety* $X$ *indexed as*

$$\{\varphi_x \mid x \in \mathcal{X}\} \cup \{\varphi_c \mid c \in \mathbf{c}\}.$$

*Assume that the following conditions hold:*

(i) $\dim(X) = N + |\mathbf{c}|$;

(ii) *the functions* $\varphi_x$ *and* $\varphi_c$ *generate the coordinate ring* $\mathbb{C}[X]$;

(iii) *every exchange relation in the cluster algebra* $\mathcal{A}$ *is valid in* $\mathbb{C}[X]$.

*Then the correspondences* $x \mapsto \varphi_x$ *and* $c \mapsto \varphi_c$ *extend uniquely to an algebra isomorphism of the cluster algebra* $\mathcal{A}$ *and* $\mathbb{C}[X]$.

$k=2$. all cluster var are Plücker coordinates

$k \geq 3$, some cluster var are not Plücker coordinates.

$Gr(3,6)$, $Gr(3,7)$, $Gr(3,8)$ finite type.
$Gr(2,n)$

Hartogs 定理: 一个函数 $f$ 在开集 $U \subset X$ 上正则 且 $U$ 之补集的余维数 $\geq 2$. 则 $f$ 可以延拓到 $X$ 上正则.

$U = \mathbb{C}^2 \setminus \{0\}$, 若 $f$ 在 $U$ 上正则. 那么 $f$ 在 $\mathbb{C}$ 上正则.

PROPOSITION 5 (Postnikov). *Let $\pi_{k,n}$ be the Grassmann permutation.*
(1) *The number of even regions in a Postnikov arrangement for $\pi_{k,n}$ is $k(n-k)+1$.*
(2) *Each even region is labelled by exactly $k$ distinct indices from $[1 \dots n]$.*

(3) *Every $k$-subset in $[1 \dots n]$ occurs as the labelling set of an even cell in some $\pi_{k,n}$-diagram.*

Laurent phenomena.

any cluster var can be expressed as a Laurent poly of the initial seed

- $\varphi_x$ is Laurent poly of intitial seed $c$

$U =$ $A_{k,n}$ 中所有 $\Delta^k \neq 0$ 的点.

$\forall$ any $k$-subset $Iij$, $|I| = k-2$

$\hookrightarrow I_{st}$

$U_{I_{ij}}$: 把 $I_{ij} \to I_{st}$ 后，所有的 $\Delta^k \neq 0$.

$$\mathcal{U} = U \bigcup_{I_{ij}} U_{I_{ij}}$$

$\Delta^{I_{ij}}$ 与 $\Delta^{I_{st}}$ 互素，故补集的余维数 $\geq 2$.

$\varphi_x$ 可以表示成 $U$ 中 cluster var 的 Laurent poly.

$\varphi_x$ 也可以表示成 $U_{I_{ij}}$ 中的 cluster var 的 Laurent poly.

$\therefore$ $\varphi_x$ 在 $Gr(k,n)$ 上正则.

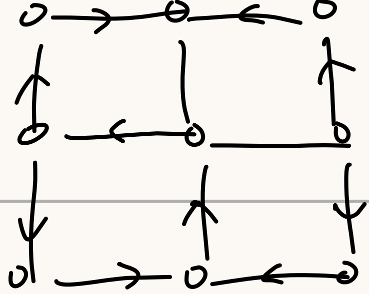
4. finite type.

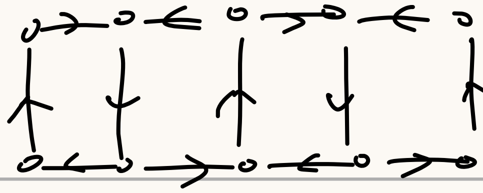
THEOREM 5. *The Grassmannians $\mathbb{G}(3,6)$, $\mathbb{G}(3,7)$, and $\mathbb{G}(3,8)$ are the only $\mathbb{G}(k,n)$, within the range $2 < k \leqslant \frac{1}{2}n$, whose homogeneous coordinate rings are cluster algebras of finite type.*

Proposition 8. [Fomin-Zelevinsky]

finite $\Longleftrightarrow$ quiver avoiding affine Dynkin diagram.

by the construction of $A_{k,n}$.

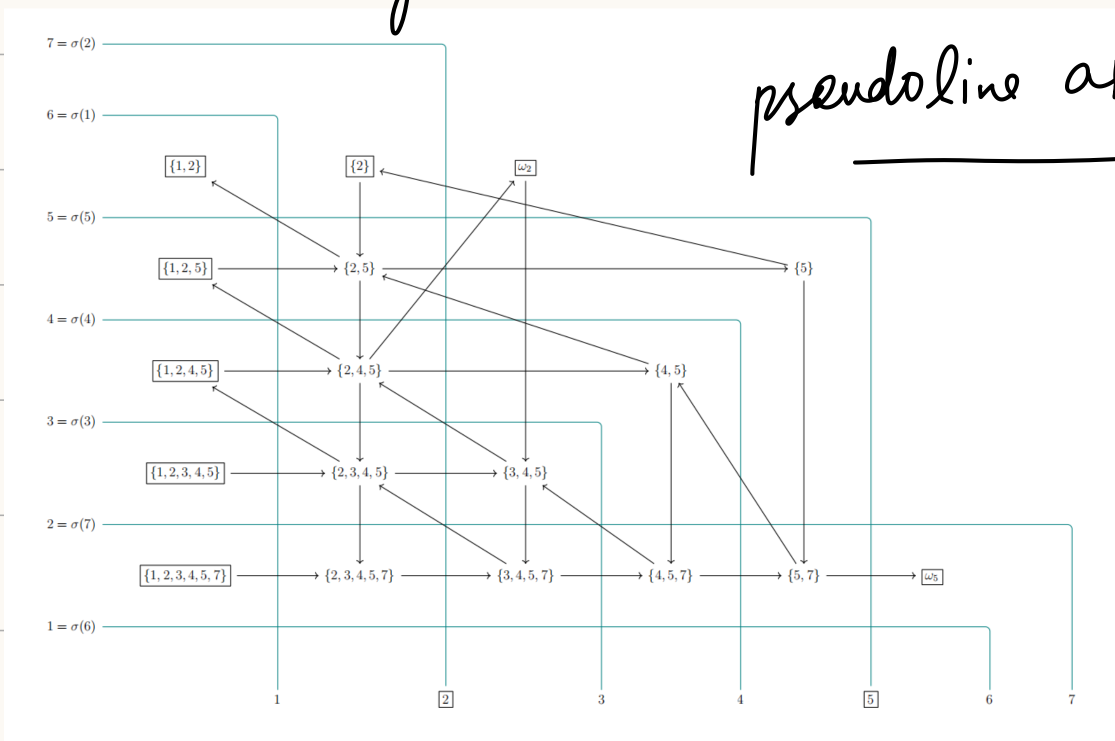




flag variety GLS'08

partial flag variety $Fl_{d_1,d_2,\cdots,d_k;n}$

Li Jianrong




pseudoline arrangement

$Fl_{2,5;7}$ $\pi = 6734512$

Richardon variety.

Leclerc 2016

preprojective category

positroid variety

Galashin-Lam 2019.

Le-diagram

Serhiyenko – Mellisa S. Bennett. 2024

braid variety JAMS. Ivent.

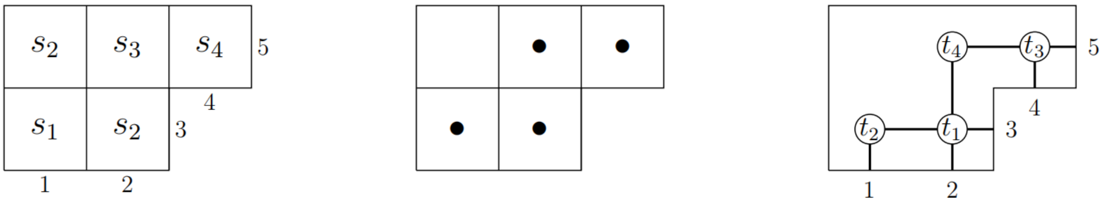



FIGURE 1. The Young diagram $\lambda$, Le-diagram $D$, and graph $G(D)$ corresponding to $(v, w) = (s_2, s_2s_1s_4s_3s_2)$.

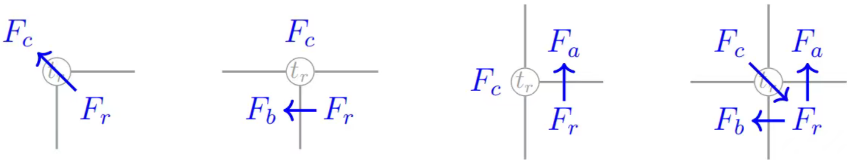



FIGURE 3. Local rules for constructing a quiver from a Le-diagram.

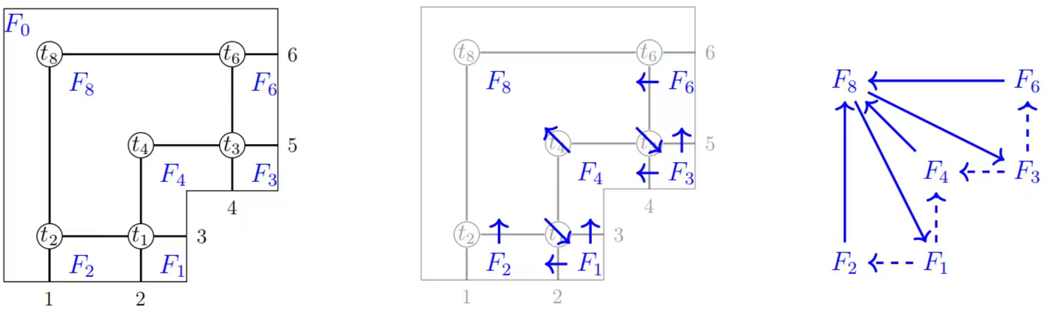



FIGURE 4. Constructing a quiver $Q_D$ from a Le-diagram $D$. See Example $\boxed{1.1}$.