Quantum Bruhat graphs and tilted Richardson varieties

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Outline

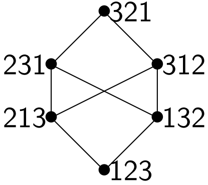
- Background on Schubert varieties and related varieties.
- Quantum Bruhat graphs and tilted Bruhat orders.
- Tilted Richardson varieties.
- Geometry of tilted Richardson varieties.

The symmetric group

- Let S_n be the symmetric group of permutations, presented as

$$S_n = \left\langle s_1, s_2, \ldots, s_{n-1} \;\middle|\; \begin{array}{r} s_i^2 = 1 \text{ for } i = 1, \ldots, n{-}1 \\ s_is_j = s_js_i \text{ if } |i-j| \geq 2 \\ s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \text{ for } i = 1, \ldots, n{-}2 \end{array} \right\rangle$$

  where $s_i = (i\ i{+}1)$ is called a **simple transposition**.
- The **Coxeter length** of $w \in S_n$ is the smallest $\ell = \ell(w)$ such that $w = s_{i_1} \cdots s_{i_\ell}$ is a product of ℓ simple transpositions.
- The **reflections** are $T = \{t_{ij} := (i\ j) \mid i < j\}$.
- The **Bruhat order** is generated by $w < wt_{ij}$ if $\ell(w) < \ell(wt_{ij})$.



Schubert calculus

- Hilbert's fifteenth problem
- counting problems of projective geometry
- study cohomology theories

The flag variety is

$$\begin{aligned} \mathrm{Fl}(\mathbb{C}^n) &= \{\emptyset = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{C}^n \mid \dim V_i = i\} \\ &= \mathrm{GL}(\mathbb{C}^n)/B \end{aligned}$$

where B is the Borel subgroup of upper triangular matrices.

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \text{ where } V_i = \mathrm{span}(v_1, \ldots, v_i).$$

Schubert calculus: the geometry

- The flag variety admits a **Bruhat decomposition**

$$\mathrm{Fl}(\mathbb{C}^n) = \bigsqcup_{w \in S_n} \Omega_w$$

  into **open Schubert cells**.
- The **Schubert variety** is $X_w := \overline{\Omega_w}$.
- $X_u \subset X_w$ if and only if $u \leq w$ in the Bruhat order.
- The **Schubert classes** $\sigma_w := [X_w]$'s form a linear basis of

$$H^*(\mathrm{Fl}(\mathbb{C}^n), \mathbb{Z}) \simeq \mathbb{Z}[x_1, \ldots, x_n]/\mathrm{Sym}^+.$$

- Major open problem: what are the structure constants?
- Can we find a combinatorial interpretation of $c_{u,v}^w \in \mathbb{Z}_{\geq 0}$?

$$\sigma_u \cup \sigma_v = \sum_{w \in S_n} c_{u,v}^w \sigma_w.$$

Schubert calculus: the quantum cohomology ring

- The **quantum cohomology ring** has a linear isomorphism

$$QH^*(\mathrm{Fl}, \mathbb{Z}) \simeq H^*(\mathrm{Fl}, \mathbb{Z}) \otimes \mathbb{Z}[q_1, \ldots, q_{n-1}].$$

- This is a free $\mathbb{Z}[q]$-module generated by $\{\sigma_w \mid w \in S_n\}$.
- The structure constant is called the **Gromov-Witten invariant**, which counts the number of degree d rational curves passing through Schubert varieties $X_u, X_v, X_{w_0 w}$, in general position:

$$\sigma_u \star \sigma_v = \sum_{w \in S_n} \langle \sigma_u, \sigma_v, \sigma_{w_0 w} \rangle_d \, q^d \sigma_w$$

  where $d = (d_1, \ldots, d_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$ and $q^d := q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$.
- Question: what weights q^d appear in the quantum product?
- In particular, what's the **minimal** such q^d? [Fulton-Woodward '04, Postnikov '05, Buch-Chung-Li-Mihalcea '20, Shifler '22]

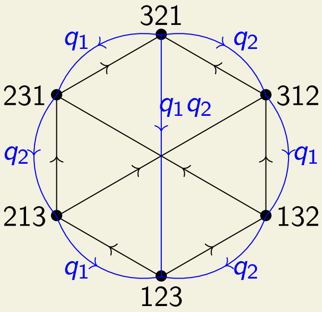
The quantum Bruhat graph

Definition (Brenti-Fomin-Postnikov '99)

The **quantum Bruhat graph** Γ_n is a weighted directed graph on S_n with the following two types of edges:

$$\begin{cases} w \to wt_{ij} \text{ of weight } 1 & \text{if } \ell(wt_{ij}) = \ell(w) + 1, \\ w \to wt_{ij} \text{ of weight } q_{ij} := q_i \cdots q_{j-1} & \text{if } \ell(wt_{ij}) = \ell(w) + 1 - 2(j-i) \end{cases}$$

The quantum Bruhat graph on S_3



The quantum Monk's rule

The quantum Bruhat graph is inspired by the quantum Monk's rule.

Theorem (Quantum Monk's rule)

Define operators $\{T_{ij} \mid 1 \leq i < j \leq n\}$ *that act* $\mathbb{Z}[q]$*-linearly on* $QH^*(\mathrm{Fl}, \mathbb{Z})$ *as*

$$T_{ij}: \ \sigma_w \mapsto \begin{cases} \sigma_{wt_{ij}} & \text{if } \ell(wt_{ij}) = \ell(w) + 1 \\ q_{ij}\sigma_{wt_{ij}} & \text{if } \ell(wt_{ij}) = \ell(w) + 1 - 2(j-i) \\ 0 & \text{otherwise} \end{cases}.$$

Then we have

$$\sigma_{s_r} \star \sigma_w = \sum_{i \leq r < j} T_{ij}\sigma_w.$$

Philosophy: Monk's rule can solve everything.

Combinatorics of the quantum Bruhat graph

For a path $P : u_0 \to u_1 \to \cdots \to u_k$, it's weight is

$$\mathrm{wt}(P) := \mathrm{wt}(u_0, u_1) \cdots \mathrm{wt}(u_{k-1}, u_k).$$

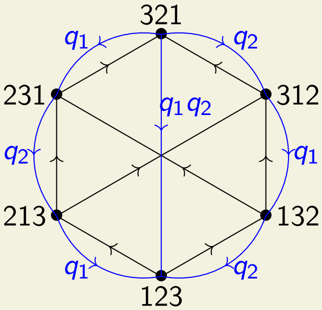
Theorem (Postnikov '04)

There is a unique minimal q^d that appears in $\sigma_u \star \sigma_{w_0 v}$. Such q^d is the weight of any shortest path in quantum Bruhat graph from u to v. Moreover, the weight of any path from u to v is divisible by q^d.

$$\{\text{minimal weight paths}\} = \{\text{shortest length paths}\}.$$

Combinatorics of the quantum Bruhat graph

The quantum Bruhat graph on S_3



Let $u = 231$ and $v = 123$.
There are two shortest paths from u to v, with $q^{d_{\min}} = q_1 q_2$.

A Simple Formula for $q^{d_{\min}}$

Theorem (Gao-Gao-G. '23, Buch-Chung-Li-Mihalcea '20)

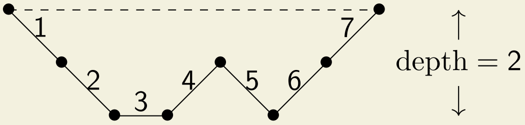
For $u, v \in S_n$, the minimal quantum degree $q^{d_{\min}}$ in $\sigma_u \star \sigma_{w_0 v}$ is

$$\prod_{k=1}^{n-1} q_i^{\mathrm{depth}(u[k],v[k])}.$$

Here $u[k] := \{u(1), \ldots, u(k)\}$.

Example of the minimal weight

For $u = 4637521, v = 5312467$ and $k = 4$, $u[k] = \{3, 4, 6, 7\}$ and $v[k] = \{1, 2, 3, 5\}$. We have $\mathrm{depth}(u[k], v[k]) = 2$.



This contributes a factor of q_4^2.

Tilted Bruhat Order

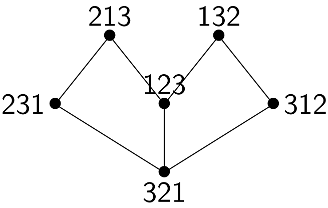
Definition (Brenti-Fomin-Postnikov '99)

For $w \in S_n$, the **tilted Bruhat order** D_w is the graded partial order "$\leq_w$" on S_n such that $u \leq_w v$ if there is a shortest path from w to v on quantum Bruhat graph passing through u.

Equivalently, let $\ell(u, v)$ be the length function on Γ_n, then

$$u \leq_w v \iff \ell(w, u) + \ell(u, v) = \ell(w, v).$$

Note that $\leq_{\mathrm{id}}$ is the usual **Bruhat order**.



An Ehresmann-like Criterion for tilted Bruhat orders

For $a \in [n]$, let $\leq_a$ be the **shifted Gale** order of $[n]$ where

$$a <_a a+1 <_a \cdots <_a n <_a 1 <_a \cdots <_a a-1.$$

For $I = \{i_1 \leq_a \cdots \leq_a i_k\}, J = \{j_1 \leq_a \cdots \leq_a j_k\} \in \binom{[n]}{k}$, we say $I \leq_a J$ if

$$i_m \leq_a j_m \text{ for all } m \in [k].$$

Theorem (Gao-Gao-G. '23)

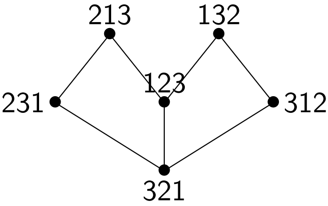
$u \leq_w v$ if there exists $\mathbf{a} = (a_1, \ldots, a_{n-1})$ such that for all $k \in [n-1]$,

$$w[k] \leq_{a_k} u[k] \leq_{a_k} v[k].$$

Equivalently, for all $\mathbf{a} = (a_1, \ldots, a_{n-1})$ such that $w[k] \leq_{a_k} v[k]$,

$$w[k] \leq_{a_k} u[k] \leq_{a_k} v[k].$$

Tilted Bruhat interval



Example of the Ehresmann criterion

Let $u = 321$, $v = 213$, then we can take $a_1 \in \{3\}$ and $a_2 \in \{2, 3\}$.
Check that $\{3\} \leq_3 \{1\} \leq_3 \{2\}$ and $\{3, 2\} \leq_{a_2} \{1, 2\} \leq_{a_2} \{2, 1\}$.
This means $123 \in [321, 213]$.

For $u \leq_x v$, define the **tilted Bruhat interval**

$$[u, v]_x := \{w \in S_n : u \leq_x w \leq_x v\}.$$

This is independent of x as long as $u \leq_x v$.
So we will choose $x = u$ and denote the interval as $[u, v]$.

Schubert varieties via rank conditions

Given a matrix M, define the south-west **rank matrix**

$$\mathrm{rk}^{SW}_{i,j}(M) := \text{ rank of the } i \times j \text{ submatrix in the SW corner of } M.$$

For permutation $v = 3142$, the rank matrix is

$$v = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathrm{rk}^{SW}_{i,j}(v) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Definition

Let $v \in S_n$. The corresponding **Schubert variety/cell** is

$$X_v := \left\{ M \in \mathrm{GL_n} \middle| \mathrm{rk}^{SW}_{i,j}(M) \leq \mathrm{rk}^{SW}_{i,j}(v) \right\} / B,$$

$$X_v^\circ := \left\{ M \in \mathrm{GL_n} \middle| \mathrm{rk}^{SW}_{i,j}(M) = \mathrm{rk}^{SW}_{i,j}(v) \right\} / B.$$

If we replace south-west (SW) with north-west (NW), we obtain the **opposite** Schubert variety/cell X^u and $(X^u)^\circ$.

Richardson varieties

For $u \le v$ in the Bruhat order, the **Richardson variety/cell** is a subvariety of Fl_n, defined as

$$\mathcal{R}_{u,v} = X^u \cap X_v \text{ and } \mathcal{R}^\circ_{u,v} = (X^u)^\circ \cap X^\circ_v.$$

In terms of rank,

$$\mathcal{R}_{u,v} := \left\{ M \in \mathrm{GL_n} \,\middle|\, \begin{array}{l} \mathrm{rk}^{NW}_{i,j}(M) \le \mathrm{rk}^{NW}_{i,j}(u) \\ \mathrm{rk}^{SW}_{i,j}(M) \le \mathrm{rk}^{SW}_{i,j}(v) \end{array} \right\} / B.$$

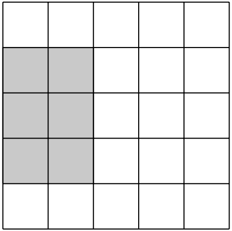
Tilted Richardson varieties

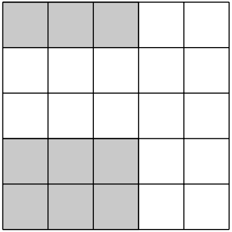
Definition (Gao-Gao-G. '23)

For $u, v \in S_n$ and a sequence $\mathbf{a} = (a_1, \ldots, a_{n-1})$ such that $u[k] \leq_{a_k} v[k]$, we define the tilted Richardson variety as

$$\mathcal{T}_{u,v,\mathbf{a}} := \left\{ M \in \mathrm{GL_n} \,\middle|\, \begin{array}{l} \mathrm{rk}_{i,j}^{\mathbf{a},NW}(M) \leq \mathrm{rk}_{i,j}^{\mathbf{a},NW}(u) \\ \mathrm{rk}_{i,j}^{\mathbf{a},SW}(M) \leq \mathrm{rk}_{i,j}^{\mathbf{a},SW}(v) \end{array} \right\} / B.$$

We define the tilted Richardson cell $\mathcal{T}^{\circ}_{u,v,\mathbf{a}}$ by replacing "$\leq$" with "$=$".



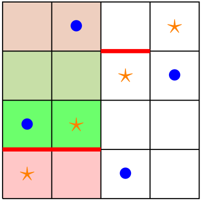


Example of tilted Richardson Varieties

Example

Consider $u = 4321, v = 3142$. We can choose $\mathbf{a} = (4,4,2)$.

Example: $j = 2$



$\mathrm{rk}^{\mathbf{a},SW}_{3,2}(M) \leq 2$

Theorem (Gao-Gao-G. '23)

$\mathcal{T}_{u,v,\mathbf{a}}$ does not depend on the choice of $\mathbf{a}$. So we denote it as $\mathcal{T}_{u,v}$.

Example

If $u \leq v$ in the Bruhat order, we can choose $\mathbf{a} = (1, \ldots, 1)$. Then we recover the Richardson variety $\mathcal{T}_{u,v} = \mathcal{R}_{u,v}$.

The tilted Richardson varieties also have many nice geometric properties:

Theorem (Gao-Gao-G. '23, '23+)

- $\mathcal{T}_{u,v}$ *is* T*-invariant and a* T*-fixed point* $w \in \mathcal{T}_{u,v} \iff w \in [u, v]$,
- $\mathcal{T}_{u,v} = \bigsqcup_{[x,y]\subseteq[u,v]} \mathcal{T}^{\circ}_{x,y}$,
- $\mathcal{T}_{u,v} = \overline{\mathcal{T}^{\circ}_{u,v}}$,
- $\dim(\mathcal{T}_{u,v}) = \dim(\mathcal{T}^{\circ}_{u,v}) =$ *length of shortest path from* u *to* v *on* Γ_n,
- $\mathcal{T}_{u,v}$ *is irreducible,*
- $[\mathcal{T}_{u,v}] = [q^{d_{min}}]\sigma_u \star \sigma_{w_0 v} = \sum_w \langle \sigma_u, \sigma_{w_0 v}, \sigma_{w_0 w} \rangle_{d_{min}} \sigma_w \in H^*(\mathrm{Fl}_n)$.

Curve neighborhoods

Recall that the Gromov-Witten invariant counts the number of degree d rational curves passing through three Schubert varieties in general position. It is also natural to study all such curves.

Definition (Buch-Chaput-Mihalcea-Perrin '13)

For permutations $u, v \in S_n$, the **two-point curve neighborhood** $\Gamma_d(X^u, X_v)$ is the union of degree d rational curves that passes through both Schubert varieties X^u and X_v in Fl_n.

Theorem (Gao-Gao-G. '23+)

For $u, v \in S_n$, $\mathcal{T}_{u,v} = \Gamma_{d_{min}}(X^u, X_v)$.

Deodhar decomposition

For $u \leq v$ in the Bruhat order, the **Kazhdan-Lusztig R polynomial** is

$$R_{u,v}(q) = \#\mathcal{R}^{\circ}_{u,v}(\mathbb{F}_q).$$

Conjecture (Combinatorial invariance problem: Lusztig '83, Dyer '87)

Across all Coxeter groups, $R_{u,v}(q)$ only depends on the combinatorial type of the Bruhat interval $[u, v]$.

In order to understand $R_{u,v}(q)$, Deodhar ('85) introduced the **Deodhar decomposition**, which decomposes $\mathcal{R}^{\circ}_{u,v}$ into simple pieces that are isomorphic to $\mathbb{C}^a \times (\mathbb{C}^*)^b$.

$$\mathcal{R}^{\circ}_{u,v} = \bigsqcup_{\alpha} \mathbb{C}^a \times (\mathbb{C}^*)^b \implies R_{u,v}(q) = \sum_{\alpha} q^a (q-1)^b.$$

Deodhar decomposition

Definition (Marsh-Rietsch '03)

Fix a reduced word $\mathbf{v} = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of v. A **distinguished subword** for u is $\mathbf{u} = u_1 \cdots u_\ell$ where each $u_k \in \{1, s_{i_k}, s_{i_k}\}$ such that

$$u_k = \begin{cases} 1 \text{ or } s_{i_k}, & \text{if } \ell(u_1 \cdots u_{k-1} s_{i_k}) > \ell(u_1 \cdots u_{k-1}), \\ s_{i_k}, & \text{if } \ell(u_1 \cdots u_{k-1} s_{i_k}) < \ell(u_1 \cdots u_{k-1}). \end{cases}$$

and their product is u. We denote $\mathbf{u} \prec \mathbf{v}$.

Example

If $\mathbf{v} = s_1 s_2 s_1$, there are two distinguished subwords for $u = \mathrm{id}$:

$$\mathbf{u} = 111 \qquad\qquad \mathbf{u} = s_1 1 s_1$$

Deodhar decomposition

For any distinguished subword $\mathbf{u} = u_1 \cdots u_\ell \prec \mathbf{v}$, define the **Deodhar cell**

$$D_{\mathbf{u},\mathbf{v}} := \{g_1 g_2 \cdots g_\ell \cdot B\} / B \subset \mathrm{Fl}_n$$

where each g_k is an $n \times n$ matrix given by

$$g_k = \begin{cases} \phi_{i_k}\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, p \neq 0, & \text{if } u_k = 1, \\ \phi_{i_k}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \text{if } u_k = s_{i_k}, \\ \phi_{i_k}\begin{pmatrix} -m & 1 \\ -1 & 0 \end{pmatrix}, & \text{if } u_k = s_{i_k}. \end{cases}$$

Here $\phi_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & a & b & \\ & & c & d & \\ 0 & & & & \ddots \end{pmatrix}$ embeds into the i^{th} and $(i+1)^{\text{th}}$ row/column of the identity matrix.

Deodhar decomposition

Theorem (Deodhar '85)

$$\mathcal{R}^{\circ}_{u,v} = \bigsqcup_{\mathbf{u}\prec\mathbf{v}} D_{\mathbf{u},\mathbf{v}}, \text{ where each } D_{\mathbf{u},\mathbf{v}} \cong (\mathbb{C}^*)^{\#1\text{'s in } \mathbf{u}} \times \mathbb{C}^{\#{\color{red}s_{i_k}}\text{'s in } \mathbf{u}}$$

If $\mathbf{v} = s_1 s_2 s_1$, there are two distinguished subwords for $u = \mathrm{id}$:

$$\mathbf{u} = 111 \qquad\qquad \mathbf{u} = s_1 1 {\color{red}s_1}$$

$D_{111,s_1s_2s_1} =$

$$\begin{pmatrix}1&0&0\\p_1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&p_2&1\end{pmatrix}\begin{pmatrix}1&0&0\\p_3&1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\p_1+p_3&1&0\\p_2p_3&p_2&1\end{pmatrix} \cong (\mathbb{C}^*)^3.$$

$D_{s_1 1 {\color{red}s_1},s_1s_2s_1} =$

$$\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&p_2&1\end{pmatrix}\begin{pmatrix}-m_3&1&0\\-1&0&0\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\-m_3&1&0\\-p_2&0&1\end{pmatrix} \cong \mathbb{C}^* \times \mathbb{C}.$$

Deodhar decomposition (proof idea)

The proof of Deodhar decoposition uses a key recursion:

Proposition (Deodhar '85)

For any simple transposition s_i such that $vs_i < v$, there are isomorphisms

$$\mathcal{R}^\circ_{u,v} \cong \begin{cases} \mathcal{R}^\circ_{us_i,vs_i}, & \text{if } us_i < u, \\ \mathcal{R}^\circ_{u,vs_i} \times \mathbb{C}^* \sqcup \mathcal{R}^\circ_{us_i,vs_i} \times \mathbb{C}, & \text{if } us_i > u. \end{cases}$$

We can induct on $\ell(u)$ and iteratively construct the decomposition.

Proposition (Gao-Gao-G. '23)

For any simple transposition s_i such that there exists sequence $\mathbf{a}$ where $u \leq_{\mathbf{a}} v$, $vs_i <_{\mathbf{a}} v$, and $a_{i-1} = a_i = a_{i+1}$, there are isomorphisms

$$\mathcal{T}^\circ_{u,v} \cong \begin{cases} \mathcal{T}^\circ_{us_i,vs_i}, & \text{if } us_i <_{\mathbf{a}} u, \\ \mathcal{T}^\circ_{u,vs_i} \times \mathbb{C}^* \sqcup \mathcal{T}^\circ_{us_i,vs_i} \times \mathbb{C}, & \text{if } us_i >_{\mathbf{a}} u. \end{cases}$$

The tilted version serves as a building block for a Deodhar like

A Deodhar decomposition for tilted Richardson cells

Theorem (Gao-Gao-G. '23+)

For any pair of permutations $u, v \in S_n$, given a **tilted reduced word v**, *we have the following decomposition:*

$$\mathcal{T}^{\circ}_{u,v} = \bigsqcup_{\mathbf{u} \prec \mathbf{v}} D_{\mathbf{u},\mathbf{v}} \cong (\mathbb{C}^*)^{\#1\text{'s in } \mathbf{u}} \times \mathbb{C}^{\# s_{i_k}\text{'s in } \mathbf{u}}.$$

There exists a unique subword $\mathbf{u}_0 \prec_t \mathbf{v}$ called the **positive distinguished subword**, *whose corresponding tilted Deodhar cell $D_{\mathbf{u}_0,\mathbf{v}} \cong (\mathbb{C}^*)^{\ell(u,v)}$ is of maximal dimension.*

Corollary (Gao-Gao-G. '23+)

$\mathcal{T}_{u,v}$ is irreducible, and $D_{\mathbf{u}_0,\mathbf{v}} \subset \mathcal{T}_{u,v}$ is a dense subset.

Future work

<u>Direction 1:</u> further geometric properties of $\mathcal{T}_{u,v}, \mathcal{T}^{\circ}_{u,v}$.

Conjecture

The tilted Richardson cells $\mathcal{T}^{\circ}_{u,v}$'s are smooth.

<u>Direction 2:</u> total positivity.

Question

Can we find a regular CW-complex realizing tilted Bruhat intervals?

In the Bruhat order, this is done via **totally nonnegative flag varieties**.

<u>Direction 3:</u> Kazhdan-Lusztig theory.

Conjecture (Gao-Gao-G. '23+)

$R^{\mathrm{tilt}}_{u,v}(q) := \#\mathcal{T}^{\circ}_{u,v}(\mathbb{F}_q)$ *depends only on the poset structure of* $[u, v]$.

Thank you all for listening!