Separable elements in Weyl groups

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Separable permutations

A permutation $w = w_1 \ldots w_n$ is called **separable** if it avoids the patterns 2413 and 3142. The following characterization of separable permutations is well known:

Proposition

A permutation w is separable if and only if there exists an index k such that $w_1 \ldots w_k$ and $w_{k+1} \ldots w_n$ are each separable permutations and $\{w_1, \ldots, w_k\}$ is either $\{1, 2, \ldots, k\}$ or $\{n-k+1, \ldots, n\}$.

Separable permutations have appeared in a variety of contexts:

- If $p_1, \ldots, p_n$ are real polynomials with
  $$p_1(x_0) = \cdots = p_n(x_0)$$
  then the permutation relating the relative order of $p_1(x_0-\varepsilon), \ldots, p_n(x_0-\varepsilon)$ to the relative order of $p_1(x_0+\varepsilon), \ldots, p_n(x_0+\varepsilon)$ is separable.
- Testing for avoidance of separable patterns in arbitrary permutations is in P, wheras the general problem is NP-complete.
- Intervals below separable permutations in weak order are the sets of linear extensions of series-parallel posets.

The weak order and inversions

Throughout, Φ is a root system with positive roots Φ^+ and simple roots Δ. The **weak order** on a Weyl group $W = W(\Phi)$ with simple reflections $s_1, \ldots, s_n$ has cover relations
$$w \lessdot s_i w$$
whenever $\ell(w) < \ell(s_i w)$ (the length ℓ is the length of the shortest expression $w = s_{i_1} \cdots s_{i_\ell}$).

Let Φ be the root system associated to W. Then the **inversion set** is:
$$I_\Phi(w) := \{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}.$$

It is a standard fact that containment of inversion sets characterizes the weak order:
$$v \le w \iff I_\Phi(v) \subseteq I_\Phi(w).$$

Patterns and root subsystems

Billey and Postnikov [1] introduced a notion of pattern avoidance for general root systems. We say Φ' is a **subsystem** of Φ if $\Phi' = \Phi \cap U$ for some linear subspace U of span(Φ). For $w \in W(\Phi)$, we say w **contains the pattern** (w', Φ') if
$$I_\Phi(w) \cap U = I_{\Phi'}(w')$$
and we write $w|_{\Phi'} = w'$ in this case. We say w avoids (w', Φ') if it does not contain any pattern isomorphic to (w', Φ').

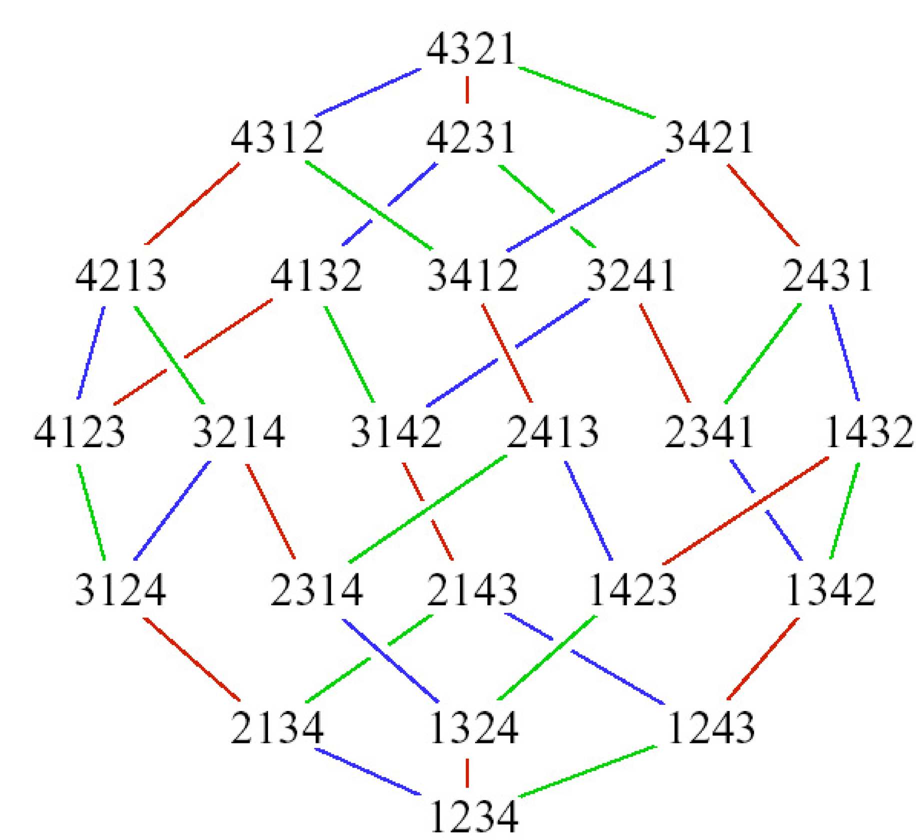

Figure 1: The left weak order on S_4. Note that the intervals below 3142 and 2413 are not rank-symmetric.

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Definition

We say an element $w \in W(\Phi)$ is *separable* if one of the following holds:

- Φ is of type A_1;
- $\Phi = \bigsqcup \Phi_i$ and $w|_{\Phi_i}$ is separable for each i;
- Φ is irreducible and there exists a *pivot* $\alpha_i \in \Delta$ such that $w|_{\Phi'} \in W(\Phi')$ is separable where Φ' is generated by $\Delta' = \Delta \setminus \{\alpha_i\}$ and such that either
  $$\{\beta \in \Phi^+ : \beta \ge \alpha_i\} \subseteq I_\Phi(w), \text{ or}$$
  $$\{\beta \in \Phi^+ : \beta \ge \alpha_i\} \cap I_\Phi(w) = \emptyset.$$

Results

Let Λ_w and V_w denote the principal order ideal and order filter generated by w in the weak order. For any poset A, let $A(q)$ denote its rank generating function. The following theorem answers an open problem of Fan Wei [2], who proved it in type A.

Theorem

Let $w \in W$ be separable. Then Λ_w and V_w are rank-symmetric and rank-unimodal and
$$\Lambda_w(q) V_w(q) = W(q).$$

When $w = w_0(J)$ for $J \subseteq S$ this result recovers the well-known fact that
$$W^J(q) W_J(q) = W(q).$$

We can also give a non-recursive characterization of separable elements, which agrees with the original definition of separable permutations.

Theorem

An element $w \in W(\Phi)$ is separable if and only if it avoids the patterns:

- 3142 and 2413 of type A_3,
- the two patterns of length two in type B_2, and
- the six patterns of lengths two, three, and four in type G_2.

Theorem

The number of separable elements in $W(\Phi)$ is twice the number of faces in the **graph associahedron** of the Dynkin diagram of Φ.

These faces are indexed by **nested sets**, we have:

- Simple combinatorial rules for going between nested sets and separable elements.
- Product formulas for $\Lambda_w(q)$ and $V_w(q)$ in terms of the corresponding nested sets.
- Combinatorial rules for obtaining the nested sets corresponding to $w_0 w$, $w w_0$, and w^{-1} in terms of the geometry of the associahedra.

Further directions

When a poset is rank-symmetric and rank-unimodal, it is often of interest to investigate the **(strong) Sperner property**: whether there are antichains larger than the middle rank of the poset. This trio of properties indicates that there is an action of $\mathfrak{sl}_2$ on the poset. This property is known for:

- The entire weak order in type A (see [3]),
- certain parabolic quotients W^J, and,
- all Λ_w and V_w for w separable which can be checked by computer.

Are Λ_w and V_w for w separable always Sperner?

References

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