Separable permutations

A permutation $w = w_1...w_n$ is called **separable** if it avoids the patterns 2413 and 3142. The following characterization of separable permutations is well known:

Proposition

A permutation w is separable if and only if there exists an index k such that $w_1...w_k$ and $w_{k+1}...w_n$ are each separable permutations and $\{w_1, ..., w_k\}$ is either $\{1, 2, ..., k\}$ or $\{n$ k+1, ..., n.

Separable permutations have appeared in a variety of contexts:

• If p_1, \ldots, p_n are real polynomials with

$$p_1(x_0) = \cdots = p_n(x_0)$$

then the permutation relating the relative order of $p_1(x_0-\varepsilon), ..., p_n(x_0-\varepsilon)$ to the relative order of $p_1(x_0 + \varepsilon), \dots, p_n(x_0 + \varepsilon)$ is separable.

- Testing for avoidance of separable patterns in arbitrary permutations is in P, wheras the general problem is *NP*-complete.
- Intervals below separable permutations in weak order are the sets of linear extensions of series-parallel posets.

The weak order and inversions

Throughout, Φ is a root system with positive roots Φ^+ and simple roots Δ . The weak order on a Weyl group W = $W(\Phi)$ with simple reflections s_1, \ldots, s_n has cover relations

$$w \lessdot s_i w$$

whenever $\ell(w) < \ell(s_i w)$ (the length ℓ is the length of the shortest expression $w = s_{i_1} \cdots s_{i_\ell}$).

Let Φ be the root system associated to W. Then the **inver**sion set is:

 $I_{\Phi}(w) := \{ \alpha \in \Phi^+ \mid w\alpha \in \Phi^- \}.$

It is a standard fact that containment of inversion sets characterizes the weak order:

$$v \le w \iff I_{\Phi}(v) \subseteq I_{\Phi}(w).$$

Separable elements in Weyl groups

Christian Gaetz and Yibo Gao

Massachusetts Institute of Technology

Patterns and root subsystems

Billey and Postnikov [1] introduced a notion of pattern avoidance for general root systems. We say Φ' is a **subsystem** of Φ if $\Phi' = \Phi \cap U$ for some linear subspace U of span (Φ) . For $w \in W(\Phi)$, we say w contains the pattern (w', Φ') if

$$I_{\Phi}(w) \cap U = I_{\Phi'}(w')$$

and we write $w|_{\Phi'} = w'$ in this case. We say w avoids (w', Φ') if it does not contain any pattern isomorphic to (w', Φ') .



Figure 1: The left weak order on S_4 . Note that the intervals below 3142 and 2413 are not rank-symmetric.

Separable elements in Weyl groups

Definition

We say an element $w \in W(\Phi)$ is *separable* if one of the following holds:

- Φ is of type A_1 ;
- $\Phi = \bigoplus \Phi_i$ and $w|_{\Phi_i}$ is separable for each i;
- Φ is irreducible and there exists a *pivot* $\alpha_i \in \Delta$ such that $w|_{\Phi'} \in W(\Phi')$ is separable where Φ' is generated by $\Delta' = \Delta \setminus \{\alpha_i\}$ and such that either

$$\{\beta \in \Phi^+ : \beta \ge \alpha_i\} \subseteq I_{\Phi}(w), \text{ or} \\ \{\beta \in \Phi^+ : \beta \ge \alpha_i\} \cap I_{\Phi}(w) = \emptyset.$$











Results

Let Λ_w and V_w denote the principal order ideal and order filter generated by w in the weak order. For any poset A, let A(q)denote its rank generating function. The following theorem answers an open problem of Fan Wei [2], who proved it in type

Theorem

Let $w \in W$ be separable. Then Λ_w and V_w are ranksymmetric and rank-unimodal and

 $\Lambda_w(q)V_w(q) = W(q).$

When $w = w_0(J)$ for $J \subseteq S$ this result recovers the well-known fact that

$$W^J(q)W_J(q) = W(q).$$

We can also give a non-recursive characterization of separable elements, which agrees with the original definition of separable permutations.

Theorem

An element $w \in W(\Phi)$ is separable if and only if it avoids the patterns:

- 3142 and 2413 of type A_3 ,
- the two patterns of length two in type B_2 , and
- the six patterns of lengths two, three, and four in type
- G_2 .

Theorem

The number of separable elements in $W(\Phi)$ is twice the number of faces in the graph associahedron of the Dynkin diagram of Φ .

These faces are indexed by **nested sets**, we have:

- Simple combinatorial rules for going between nested sets and separable elements.
- Product formulas for $\Lambda_w(q)$ and $V_w(q)$ in terms of the corresponding nested sets.
- Combinatorial rules for obtaining the nested sets corresponding to $w_0 w, w w_0$, and w^{-1} in terms of the geometry of the associahedra.

Are Λ_w and V_w for w separable always Sperner?

- [2] Fan Wei.

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Further directions

When a poset is rank-symmetric and rank-unimodal, it is often of interest to investigate the (strong) sperner property: whether there are antichains larger than the middle rank of the poset. This trio of properties indicates that there is an action of \mathfrak{sl}_2 on the poset. This property is known for:

• The entire weak order in type A (see [3]),

• certain parabolic quotients W^J , and,

• all Λ_w and V_w for w separable which can be checked by computer.

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Contact Information

