Separable elements and splittings of Weyl groups

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Overview

1. Separable premrutations
2. Separable elements in Weyl groups
   - Definitions
   - Main properties
   - Pattern avoidance characterization
   - Faces of graph associahedra
3. Splittings of Weyl groups
   - A solution to a problem of Björner and Wachs on generalized quotients
   - A combinatorial proof of Sidorenko's inequality on linear extensions

Separable permutations

Definition

A permutation is **separable** if it avoids the patterns 3142 and 2413.

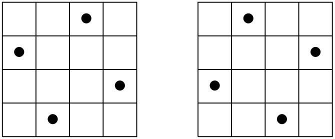


Figure: Permutations 3142 and 2413.

Lemma

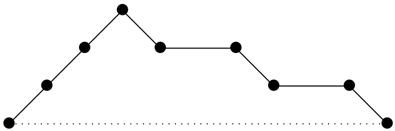
If $w \in \mathfrak{S}_n$ is separable, then there exists $1 < m < n$ such that either

- *$w_1 \cdots w_m$ is a separable permutation on $\{1, \ldots, m\}$ and $w_{m+1} \cdots w_n$ is a separable permutation on $\{m+1, \ldots, n\}$;*
- *or $w_1 \cdots w_m$ is a separable permutation on $\{n-m+1, \ldots, n\}$ and $w_{m+1} \cdots w_n$ is a separable permutation on $\{1, \ldots, n-m\}$.*

Separable permutations: fun facts

Separable permutations were first introduced by Bose, Buss and Lubiw in 1998:

- Testing for avoidance of a separable permutation pattern can be done in polynomial time (*NP*-complete in general),
- They are counted by Schröder numbers



- Appear in the theory of *pop-stack* sorting,
- If a collection of distinct real polynomials all have equal values at some number x, then the permutation that describes how the numerical ordering of the polynomials changes at x is separable, and every separable permutation can be realized in this way.
- Some interesting combinatorics related to the **weak order**.

Notations on ranked posets

Let P be a finite ranked poset with rank decomposition $P_0 \sqcup P_1 \sqcup \cdots \sqcup P_r$. We say that P is

- rank symmetric if $|P_i| = |P_{r-i}|$ for all i,
- rank unimodal if there exists m such that $|P_0| \leq |P_1| \leq \cdots \leq |P_m| \geq \cdots \geq |P_{r-1}| \geq |P_r|$.

For $x \in P$, let

- $V_x := \{y \in P : y \geq x\}$ be the principal upper order ideal at x,
- $\Lambda_x := \{y \in P : y \leq x\}$ be the principal lower order ideal at x.

Let

$$F(P) = P(q) := \sum_{x \in P} q^{\mathrm{rk}(x)}$$

be the rank generating function of P.

Background on the weak (Bruhat) order

The right weak (Bruhat) order R_n is generated by

$$w \lessdot_R ws_i \quad \text{if } \ell(ws_i) = \ell(w) + 1, \text{ where } s_i = (i, i+1).$$

The left weak (Bruhat) order L_n is generated by

$$w \lessdot_L s_i w \quad \text{if } \ell(s_i w) = \ell(w) + 1, \text{ where } s_i = (i, i+1).$$

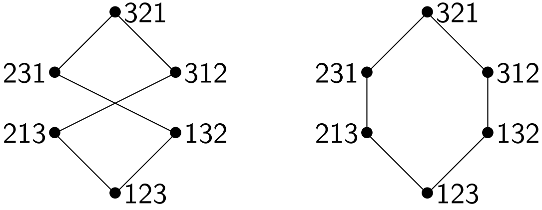


Figure: The left weak order and the right weak order on $\mathfrak{S}_3$.

Results by Fan Wei

Theorem (Wei 2012)

Let $\pi \in \mathfrak{S}_n$ be a separable permutation. Then both Λ_π and V_π are rank symmetric and rank unimodal. Moreover, $\Lambda_\pi(q)V_\pi(q) = \mathfrak{S}_n(q)$.

Her proof relies on the following lemma.

Lemma (Wei 2012)

Let $\pi = uv$ as words where u and v are separable. Then

- *if $u \in \mathfrak{S}_{1,\ldots,m}, v \in \mathfrak{S}_{m+1,\ldots,n}$, $\Lambda_\pi(q) = \Lambda_u(q)\Lambda_v(q)$ and $V_\pi(q) = V_u(q)V_v(q)\begin{bmatrix} n \\ m \end{bmatrix}_q$;*
- *if $u \in \mathfrak{S}_{m+1,\ldots,n}, v \in \mathfrak{S}_{1,\ldots,m}$, $\Lambda_\pi(q) = \Lambda_u(q)\Lambda_v(q)\begin{bmatrix} n \\ m \end{bmatrix}_q$ and $V_\pi(q) = V_u(q)V_v(q)$.*

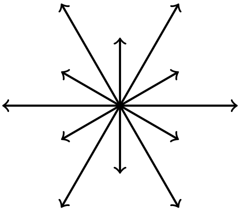
We will be generalizing these results to other types.

Root systems and Weyl groups

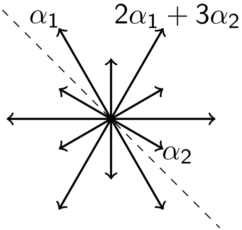
Definition (Root system)

Let $E=\mathbb{R}^n$. A root system $\Phi\subset E$ is a finite set of vectors, such that

- Φ spans E;
- for $\alpha\in\Phi$, $k\alpha\in\Phi$ iff $k\in\{\pm 1\}$;
- for $\alpha,\beta\in\Phi$, $2(\alpha,\beta)/(\alpha,\alpha)\in\mathbb{Z}$;
- for $\alpha,\beta\in\Phi$,
$$\sigma_\alpha(\beta):=\beta-\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha\in\Phi.$$



Root systems and Weyl groups



Picking a generic hyperplane partitions Φ into **positive roots** Φ^+ and negative roots Φ^-. This determines a unique set Δ of **simple roots** such that

- $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is a basis for E;
- every $\alpha \in \Phi^+$ is written as $\sum_{i=1}^n c_i\alpha_i$ where $c_i \in \mathbb{Z}_{\geq 0}$ $\forall i$.

Root systems and Weyl groups

We say Φ is **irreducible** if it cannot be partitioned into $\Phi' \sqcup \Phi''$ such that $(\alpha, \beta) = 0$ for all $\alpha \in \Phi'$ and $\beta \in \Phi''$.

Irreducible root systems can be classified using Dynkin diagrams.

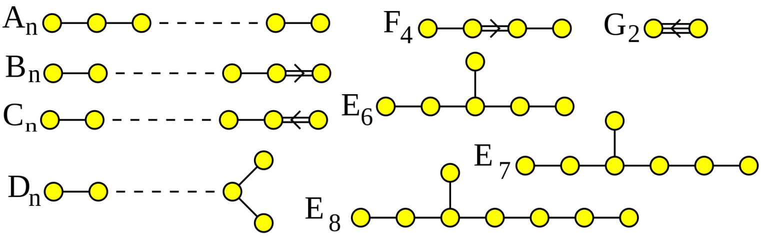


Figure: Irreducible root systems (Wikipedia)

Root systems and Weyl groups

The **Weyl group** $W = W(\Phi)$ that corresponds to Φ is a finite subgroup of $\mathrm{GL}(E)$ generated by all reflections across roots σ_α, for $\alpha \in \Phi$, or equivalently, by $s_i := \sigma_{\alpha_i}$ for $\alpha_i \in \Delta$.

Fix $\Delta \subset \Phi^+ \subset \Phi$ as above.

For $w \in W$, its **Coxeter length** $\ell(w)$ is defined to be the smallest ℓ such that w can be written as $s_{i_1} \cdots s_{i_\ell}$.

The left weak (Bruhat) order is generated by

$$w \lessdot_L s_i w \quad \text{if } \ell(s_i w) = \ell(w) + 1, \text{ where } s_i = \sigma_{\alpha_i}, \alpha_i \in \Delta.$$

Root systems and Weyl groups

Example: root system of type A_{n-1}

- $E = \mathbb{R}^n/(1,\ldots,1)$. $\Phi = \{e_i - e_j : i \neq j\}$.
- $\Phi^+ = \{e_i - e_j : i < j\}$.
- $\Delta = \{e_i - e_{i+1} : i = 1,\ldots,n-1\}$.
- $\sigma_{e_i - e_j} : (x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n) \mapsto (x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n)$.
- $s_i = \sigma_{e_i - e_{i+1}} = (i\ i+1)$, so $W \cong \mathfrak{S}_n$.

Root systems and Weyl groups

Definition (Inversion set)

For $w \in W$, $I_\Phi(w) := \{\alpha \in \Phi^+ : w\alpha \in \Phi^-\}$.

The following proposition is well-known and useful.

Proposition

- *$I_\Phi(w)$ uniquely characterizes $w \in W$.*
- *$S \subset \Phi^+$ is the inversion set of some $w \in W$ iff S is* **biconvex***:*
  - *if $\alpha, \beta \in S$ and $\alpha + \beta \in \Phi^+$, then $\alpha + \beta \in S$;*
  - *if $\alpha, \beta \notin S$ and $\alpha + \beta \in \Phi^+$, then $\alpha + \beta \notin S$.*
- *$u \leq_L v$ in the (left) weak order iff $I_\Phi(u) \subset I_\Phi(v)$.*

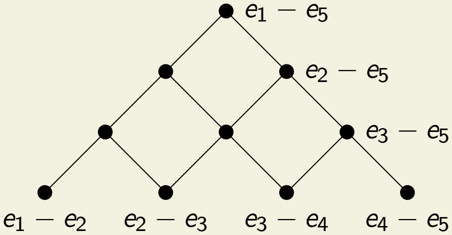
Definition (Root poset and support)

For $\alpha, \beta \in \Phi^+$, $\alpha \leq \beta$ if $\beta - \alpha$ is written as a nonnegative linear combination of simple roots. For $\alpha \in \Phi^+$, its support is defined as $\mathrm{Supp}(\alpha) := \{\alpha_i \in \Delta : \alpha_i \leq \alpha\}$.

Root systems and Weyl groups

Example: root system of type A_{n-1}

- $E = \mathbb{R}^n/(1,\dots,1)$. $\Phi = \{e_i - e_j : i \neq j\}$.
- $\Phi^+ = \{e_i - e_j : i < j\}$.
- $\Delta = \{e_i - e_{i+1} : i = 1,\dots,n-1\}$.
- $\sigma_{e_i - e_j} : (x_1,\dots,x_i,\dots,x_j,\dots,x_n) \mapsto (x_1,\dots,x_j,\dots,x_i,\dots,x_n)$.
- $s_i = \sigma_{e_i - e_{i+1}} = (i\ i+1)$, so $W \cong \mathfrak{S}_n$.
- $I_\Phi(w) = \{e_i - e_j \ : \ i < j, \text{and } w_i > w_j\}$.



A restriction map (Billey-Postnikov 2005)

- Let $E' \subset E$ be a subspace.
- $\Phi' := E' \cap \Phi$ is a root system.
  $(\Phi')^+ := E' \cap \Phi^+$ is a choice of positive roots.
- For $w \in W(\Phi)$, $I_\Phi(w)$ is biconvex.
  So $I_\Phi(w) \cap E' \subset (\Phi')^+$ is also biconvex.
- Define $w|_{\Phi'} := w'$ to be the unique $w' \in W(\Phi')$ such that $I_{\Phi'}(w') = I_\Phi(w) \cap E'$.

Example: restriction map in type A

- Let $w = 6347215 \in W(A_6)$.
- Consider $E' = \mathrm{span}(e_2 - e_4, e_4 - e_5)$. Then Φ' is of type A_2 with the set of simple roots $\Delta' = \{e_2 - e_4, e_4 - e_5\} = \{e'_1 - e'_2, e'_2 - e'_3\}$.
- Then $I_\Phi(w) \cap E' = \{e_4 - e_5, e_2 - e_5\} = \{e'_2 - e'_3, e'_1 - e'_3\}$ since $w(4) > w(5)$ and $w(2) > w(5)$.
- So $w|_{\Phi'} = 231 \in W(A_2)$.

Separable elements in Weyl groups

Definition (Gaetz and G. 2019)

Let $w \in W(\Phi)$. Then w is **separable** if one of the following holds:

- Φ is of type A_1;
- $\Phi = \bigoplus \Phi_i$ is reducible and $w|_{\Phi_i}$ is separable for all i;
- Φ is irreducible and there exists a **pivot** $\alpha_i \in \Delta$ such that $w|_{\Phi'} \in W(\Phi')$ is separable, where Φ' is generated by $\Delta \setminus \{\alpha_i\}$ and either $\{\alpha \in \Phi^+ : \alpha \geq \alpha_i\} \subset I_\Phi(w)$ or $\{\alpha \in \Phi^+ : \alpha \geq \alpha_i\} \cap I_\Phi(w) = \emptyset$.

Compare the following equivalent definition of separable permutations.

Definition

Let $w \in \mathfrak{S}_n$. Then w is separable if one of the following holds:

- $n \leq 2$;
- there exists $1 < m < n$ such that either
  - $w_1 \cdots w_m$ is a separable permutation on $\{1, \ldots, m\}$ and $w_{m+1} \cdots w_n$ is a separable permutation on $\{m+1, \ldots, n\}$;
  - or $w_1 \cdots w_m$ is a separable permutation on $\{n-m+1, \ldots, n\}$ and $w_{m+1} \cdots w_n$ is a separable permutation on $\{1, \ldots, n-m\}$.

Example (separable elements in $W(B_2)$)

$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$. $\Delta = \{\alpha_1, \alpha_2\}$. Dynkin diagram  .

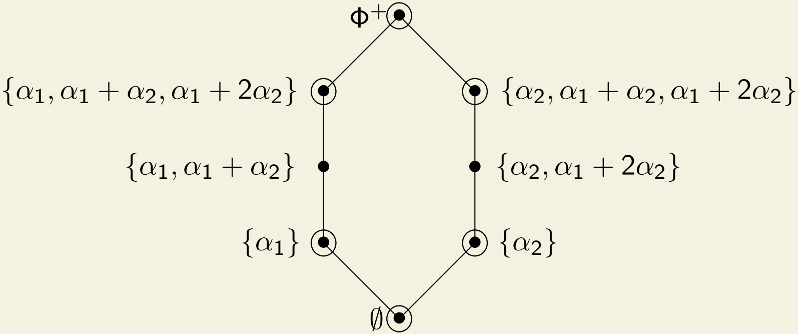


Figure: Weak order of type B_2 labeled by inversion sets, where separable elements are circled.

Properties of separable elements

The definition of separable elements, together with the following theorem, answers an open problem of Fan Wei.

Theorem (Gaetz and G. 2019)

Let $w \in W = W(\Phi)$ be separable. Then the upper order ideal V_w and the lower order ideal Λ_w in weak order are both rank-symmetric and rank-unimodal, and

$$V_w(q)\Lambda_w(q) = W(q).$$

The longest element w_0^J in the **parabolic quotient** W^J is separable. In this case we recover the well-known that that

$$W^J(q)W_J(q) = W(q),$$

where W_J is the **parabolic subgroup**.

Classification via pattern avoidance

Definition (Pattern avoidance)

We say that $w \in W(\Phi)$ avoids pattern $w' \in W(\Phi')$ if there does not exists a subspace $E' \subset E$ such that $\Phi' \simeq E' \cap \Phi$ and $w|_{\Phi'} = w'$.

Theorem (Gaetz and G. 2019)

An element $w \in W(\Phi)$ is separable if (and only if) it avoids:

- *2413 and 3142 in $W(A_3)$,*
- *two patterns of length 2 in $W(B_2)$,*
- *and six patterns of length 2,3,4 in $W(G_2)$.*

Our proof is fairly technical, type-dependent and computer-assisted.

Remark

$|W(E_8)| = 696{,}729{,}600$.

Faces of graph associahedra

Let Γ be a simple graph. The **graph associahedron** $A(\Gamma)$ is a polytope which can be defined as the Minkowski sum of coordinate simplices corresponding to the connected subgraphs of Γ.

Definition (Postnikov 2009)

A collection $\mathcal{N}$ of subsets of Γ is a **nested set** if

- for all $J \in \mathcal{N}$, the induced subgraph $\Gamma|_J$ is connected,
- for any $I, J \in \mathcal{N}$, either $I \subset J$, $J \subset I$ or $I \cap J = \emptyset$,
- for any collection of $k \geq 2$ disjoint $J_1, \ldots, J_k \subset \mathcal{N}$, then subgraph $\Gamma|_{J_1 \cup \cdots \cup J_k}$ is not connected.

Proposition (Postnikov 2009)

The poset of faces of $A(\Gamma)$ is isomorphic to the poset of nested sets on Γ which contain all connected components of Γ, ordered by reverse containment.

Faces of graph associahedra

Theorem (Gaetz and G. 2019)

Let W be a finite Weyl group whose Dynkin diagram Γ contains r connected components. Then

1. *the nested sets on Γ are in bijection with separable elements of W:*

$$\mathcal{N} \mapsto \prod_{J \in \mathcal{N}} w_0(J) =: w(\mathcal{N}),$$

   where the product is taken in the order of any linear extension

2. *the rank generating function of the intervals $[e, w(\mathcal{N})]$ is*

$$\Lambda^L_{w(\mathcal{N})}(q) = \frac{\prod_{J \in \mathcal{N}_{\mathrm{even}}} W_J(q)}{\prod_{J \in \mathcal{N}_{\mathrm{odd}}} W_J(q)}.$$

In particular, separable elements of W are in bijection with 2^r copies of faces of $A(\Gamma)$.

Faces of graph associahedra

Example: bijection from nested sets to separable elements

- Let W be a Weyl group of type A_4 with simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.
- The rank generating function for W is $W(q) = [5]!_q$.
- $\mathcal{N} = \{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \{\alpha_1, \alpha_2\}, \{\alpha_2\}, \{\alpha_4\}\}$.
- $w(\mathcal{N}) = 54321 \cdot 32145 \cdot 13245 \cdot 12354 = 35412$.
- We see that $354|12$ has a pivot at α_3. And

$$\Lambda^L_{w(\mathcal{N})}(q) = \frac{[5]!_q[2]!_q}{[3]!_q[2]!_q} = q^7 + 2q^6 + 3q^5 + 4q^4 + 4q^3 + 3q^2 + 2q + 1.$$

- $\mathcal{N}' = \{\{\alpha_1, \alpha_2\}, \{\alpha_2\}, \{\alpha_4\}\}$.
- $w(\mathcal{N}') = 32145 \cdot 13245 \cdot 12354 = 31254$. And

$$\Lambda^L_{w(\mathcal{N}')}(q) = \frac{[3]!_q[2]!_q}{[2]!_q} = q^3 + 2q^2 + 2q + 1.$$

Generalized quotients and splittings of Weyl groups

Definition (Björner and Wachs 1988)

Given a subset U of a Weyl group W, the **generalized quotient** is

$$W/U := \{w \in W \mid \ell(wu) = \ell(w) + \ell(u), \forall u \in U\}.$$

It generalizes parabolic quotients, since $W^J = W/W_J$.

Proposition (Björner and Wachs 1988)

Let $u_0 = \bigvee_{u \in U}^{R} u$. *Then* $W/U = [e, w_0 u_0^{-1}]_L$.

In finite Weyl groups, generalized quotients are just intervals in the left weak order.

Generalized quotients and splittings of Weyl groups

Definition (Björner and Wachs 1988)

A pair (X, Y) of arbitrary subsets $X, Y \subset W$ such that the multiplication map $X \times Y \to W$ sending $(x, y) \mapsto xy$ is length-additive and bijective is called a **splitting** of W.

For example, (W^J, W_J) is a splitting of W, for any $J \subset \Delta$.

Problem (Björner and Wachs 1988)

In the case $W = \mathfrak{S}_n$, for which $U \subset W$ is the multiplication map

$$W/U \times U \to W$$

a splitting of W?

This map is length-additive by definition. So the problem is asking for which U is this map a bijection?

Generalized quotients and splittings of Weyl groups

Theorem (Gaetz and G. 2019)

Let W be any finite Weyl group and $U = [e, u]_R$ with u separable, then $(W/U, U)$ is a splitting of W.

The following main theorem answers more than the problem posed by Björner and Wachs.

Theorem (Gaetz and G. 2019)

Let (X, Y) be an arbitrary splitting of $W = \mathfrak{S}_n$, then $X = W/Y$ and $Y = [e, u]_R$ with u separable.

Conjecture (Gaetz and G. 2019)

Let W be any finite Weyl group and let $U = [e, u]_R \subset W$. Then $(W/U, U)$ is a splitting of W if and only if u is separable.

The surjection theorem

Let $u \in \mathfrak{S}_n$ and $U = [e, u]_R$. We saw that the multiplication map

$$W/U \times U \to W$$

is bijective if and only if u is separable.

Theorem (Gaetz and G. 2019)

For any $u \in \mathfrak{S}_n$, the multiplication map

$$W/U \times U \to \mathfrak{S}_n$$

is surjective.

Sidorenko's inequality on linear extensions

A *linear extension* of a poset $P = \{p_1, ..., p_n\}$ is an order preserving bijection

$$\lambda : P \to [n].$$

The number of linear extensions of P is denoted $e(P)$.

We are interested in **two-dimensional posets**. These are partial orders P_u on $\{p_1, ..., p_n\}$ for $u \in \mathfrak{S}_n$ such that

$$p_i \leq p_j \iff i \leq j \text{ and } u^{-1}(i) \leq u^{-1}(j).$$

Proposition (Björner and Wachs 1991)

The linear extensions of P_u are exactly the elements of $[e, u]_R$.

Sidorenko's inequality on linear extensions

The **complement** $\overline{P}$ of a poset P has complementary comparability graph to that of P. The choice of a complement is not unique, but $e(\overline{P})$ is well-defined.

It is known that P has a complement if and only if P is two-dimensional, and P_u has a natural complement $\overline{P_u} = P_{uw_0}$.

Theorem (Sidorenko 1991)

Let P be a two-dimensional poset, then $e(P)e(\overline{P}) \geq n!$ with equality if and only if P is series-parallel.

A **series-parallel poset** is constructed from • by disjoint union and direct sum.

Sidorenko's inequality on linear extensions

Theorem (Sidorenko 1991)

Let P be a two-dimensional poset, then $e(P)e(\overline{P}) \geq n!$.

Known proofs:

- Sidorenko: uses analysis of various recurences and the Max-flow/Min-cut Theorem,
- Bollobás, Brightwell and Sidorenko: use a special case of the still-open Mahler conjecture from convex geometry and the difficult Perfect Graph Theorem.

The surjection theorem provides an explicit combinatorial proof, answering an open problem of Morales, Pak, and Panova. We also obtain a q-analog.

Theorem (Gaetz and G. 2019)

Let $u \in \mathfrak{S}_n$. Then $[e, w_0u^{-1}]_L \times [e, u]_R \to \mathfrak{S}_n$ is surjective.

The surjection theorem

Theorem (Gaetz and G. 2019)

Let $u \in \mathfrak{S}_n$. Then $[e, w_0 u^{-1}]_L \times [e, u]_R \to \mathfrak{S}_n$ is surjective.

The following reformation is my favorite.

Theorem

For any $w, \pi \in \mathfrak{S}_n$, there exists $u \in \mathfrak{S}_n$, such that $u \leq_L w$, $u \leq_R \pi$, and $(wu^{-1})(u)(u^{-1}\pi)$ is a reduced expression.

Here, we say $w_1 \cdots w_k$ is reduced if $\ell(w_1) + \cdots + \ell(w_k) = \ell(w_1 \cdots w_k)$.

Examples of the surjection theorem

- If $\ell(w\pi) = \ell(w) + \ell(\pi)$, i.e. $w\pi$ is reduced, we can take $u = e$.
- If $w \leq_R \pi$, we can take $u = w$.
- The choice of u may not be unique. Consider $w = \pi = 3142$. Then u can be either e or 3142.

Proof of the surjection theorem

Theorem (Gaetz and G. 2019)

For any $w, \pi \in \mathfrak{S}_n$, there exists $u \in \mathfrak{S}_n$, such that $u \leq_L w$, $u \leq_R \pi$, and $(wu^{-1})(u)(u^{-1}\pi)$ is a reduced expression.

Our main theorem relies on the following technical lemma.

Lemma

Let $w, \pi, u \in \mathfrak{S}_n$, such that $u \leq_L w$, $u \leq_R \pi$ and $(wu^{-1})(u)(u^{-1}\pi)$ is not reduced, then there exists $u' >_S u$ such that $u' \leq_L w$ and $u' \leq_R \pi$.

Conjecture

The above lemma (theorem) is true for any finite Weyl groups.

Proof of the surjection theorem

Lemma

Let $w, \pi, u \in \mathfrak{S}_n$, such that $u \leq_L w$, $u \leq_R \pi$ and $(wu^{-1})(u)(u^{-1}\pi)$ is not reduced, then there exists $u' >_S u$ such that $u' \leq_L w$ and $u' \leq_R \pi$.

Our proof relies on the use of wiring diagrams.

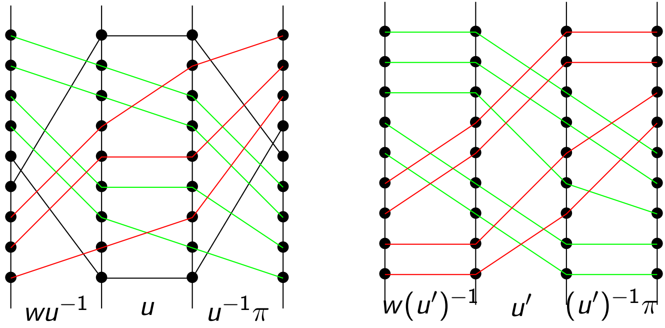


Figure: The initial wiring diagram (left) and the construction of u' (right).

Thanks

Our thanks to: Alex Postnikov, Anders Björner, Vic Reiner, Richard Stanley, and Igor Pak.

Thank you for listening!