Analysis I 2026

Guo Chuan Thiang

1 Lectures 1–2

- Calculus/analysis is closely linked to geometry, mechanics, physics. Non-rigourous version suffices for many scientific and engineering applications.
  - More advanced and powerful analysis tools, e.g., complex analysis, Fourier and harmonic analysis, functional analysis, PDEs, are needed in more modern applications, and also in pure mathematics problems (usually close to geometry).
  - When used without fundamental understanding, powerful tools can become dangerous and lead to all sorts of nonsensical conclusions. Tricky examples of Zeno paradox, geometric series, harmonic series were sketched.
  - Even if "real world" systems involve finitely many steps, elements, etc., the concepts of "infinity" and "limits" studied in analysis are extremely relevant. They help us to reason carefully and confidently about a whole class of (sufficiently large) "finite" systems.
  - The "infinitesimally small" concept has likewise been a powerful concept for understanding mechanics.
- Usually, natural numbers are informally thought of as forming a set,

$$\mathbb{N} := \{0, 1, 2, \ldots\}.$$

"Set" is collection of "objects" under discussion. Basic property is set membership, $x \in X$, meaning "object x is an element of set X". For most purposes, we only need to manipulate sets naïvely.

- Allowing, e.g., for "set of all sets" leads to logical paradoxes (statements which are both true and false). So mathematicians developed *axiomatic set theory* to impose constraints forbidding creation of certain sets. Implicitly, we work with "ZFC axioms" (used generally by default in most of mathematics). We do not need to know precisely the presentation of this axiom system. Here are some of the basic things we are allowed to do with sets:
  - Sets A and B are *equal* if every element of A is an element of B, and vice versa.
  - Empty set $\emptyset$ exists, characterized by property
    $$\forall\, x,\ x \notin \emptyset.$$
    Sets can themselves be objects in another set, e.g.,
    $$\{\emptyset\}, \qquad \{\emptyset, \{\emptyset\}\}.$$
  - Unions and intersections are sets. These operations are commutative, associative operations, satisfying distributivity,
    $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
    $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
    Notation: A *disjoint union* (meaning $A \cap B = \emptyset$) is denoted $A \sqcup B$.
  - If background set X is fixed, the *complement* of a subset $A \subseteq X$ in X is the subset
    $$A^c \equiv X \setminus A := \{x \in X \,:\, x \notin A\}.$$
    De Morgan laws:
    $$(A \cup B)^c = A^c \cap B^c,$$
    $$(A \cap B)^c = A^c \cup B^c.$$
- Function $f : X \to Y$ assigns each element $x \in X$ to an element $f(x) \in Y$. The sets X, Y are called the *domain* and *domain* of f, respectively. The *range* of f is
  $$\text{Range}(f) = \{f(x) \,:\, x \in X\} \subseteq Y.$$

We defined *injective, surjective, bijective* function. In the last case, the *inverse function* $f^{-1} : Y \to X$ satisfying

$$f \circ f^{-1} = \mathrm{id}_Y, \qquad f^{-1} \circ f = \mathrm{id}_X,$$

exists. For a general function f and a general subset $A \subseteq Y$, we write

$$f^{-1}(A) := \{x \in X \,:\, f(x) \in A\} \subseteq X.$$

- Set theory axioms will include some "axiom of infinity". Its practical consequence is the existence of a set $\mathbb{N}$, equipped with a *successor function* $S : \mathbb{N} \to \mathbb{N}$, satisfying the *Peano axioms*:

  P0. $\mathbb{N}$ contains an initial element, denoted by 0.

  P1. $\forall n \in \mathbb{N}, S(n) \neq 0$.

  P2. S is injective.

  P3. If $A \subset \mathbb{N}$ satisfies

  $$\begin{cases} 0 \in A, \\ n \in A \;\Rightarrow\; S(n) \in A, \end{cases}$$

  then $A = \mathbb{N}$.

  The Peano axioms characterize $\mathbb{N}$ uniquely in a certain sense. It is best to return to what this means at a later stage of mathematical learning. For now, we simply take for granted that $\mathbb{N}$ is "the" set satisfying the above Peano axioms, and build up from this.

  The "first few" natural numbers are denoted by

  $$1 := S(0), \qquad 2 := S(1), \qquad 3 := S(2),$$

  and so on.

- For each $n \in \mathbb{N}$, let $P(n)$ be a true/false statement. Suppose we know that

  $$\begin{cases} P(0) \text{ true}, \\ \text{For each } n \in \mathbb{N}, \quad P(n) \text{ true} \Rightarrow P(S(n)) \text{ true}. \end{cases}$$

  Then the set

  $$A = \{n \in \mathbb{N} \,:\, P(n) \text{ is true}\}.$$

satisfies the condition of Axiom P3, therefore $A = \mathbb{N}$. In other words, we may deduce that $P(n)$ is true for every $n \in \mathbb{N}$. This kind of argument is *mathematical induction*.

Exercise: Use induction to prove that every nonzero natural number is the successor of some natural number.

- For sets X, Y, we can consider *ordered pairs* (x, y) where $x \in X$ and $y \in Y$. (We could write $X = X_1$, $Y = X_2$, with the subscript $2 = S(1)$ indicating that X_2 comes after X_1.) The set of ordered pairs is the *Cartesian product*,

$$X \times Y := \{(x, y) \,:\, x \in X, y \in Y\}.$$

  This generalizes to the Cartesian product $X_1 \times \cdots \times X_n$ for $n \in \mathbb{N}$.

- *Addition* of natural numbers,

$$\begin{aligned} \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \\ (m, n) &\mapsto m + n, \end{aligned}$$

  is defined recursively by the rules

$$\begin{aligned} 0 + n &:= n, & n &\in \mathbb{N}, \\ S(m) + n &:= S(m + n), & m, n &\in \mathbb{N}. \end{aligned}$$

  All cases are covered because of the earlier Exercise. Again, we omit discussion of why such recursively definitions are "allowed".

  Note that

$$1 + n = S(n).$$

  Using Peano axioms for $\mathbb{N}$, we can verify that addition satisfies commutativity, associativity, as well as

$$\forall a, b, c \in \mathbb{N}, \quad a + b = a + c \;\Rightarrow\; b = c. \qquad \text{(cancellation)}$$

- *Ordering relation* $\leq$ on $\mathbb{N}$ is defined by:

$$m \leq n \quad \text{iff} \quad \exists a \in \mathbb{N} \text{ such that } n = m + a.$$

  If $m \leq n$ and $m \neq n$, we write $m < n$.

  Using Peano axioms, we can verify that for all $a, b, c \in \mathbb{N}$,

Reflexive: $a \leq a$,

Transitive: $a \leq b$ and $b \leq c \Rightarrow a \leq c$,

Anti-symmetric: $a \leq b$ and $b \leq a \Rightarrow a = b$,

Addition preserves order: $a \leq b \Rightarrow a + c \leq b + c$.

Trichotomy: Exactly one of $a < b$, $a = b$, $b < a$ holds.

- In general, a binary *relation* R on a set X is a subset $R \subseteq X \times X$. The idea is that an ordered pair (a, b) lies in R precisely when "a is related to b"; this is denoted

$$aRb.$$

If the relation R satisfies

Reflexive: aRa,

Transitive: aRb and $bRc \Rightarrow aRc$,

Anti-symmetric: aRb and $bRa \Rightarrow a = b$,

we say that R is a *partial order* on X. If, furthermore,

$$\forall a, b \in X, \qquad \text{either } aRb \text{ or } bRa,$$

we say that R is a *total order* on X.

The ordering relation $\leq$ on $\mathbb{N}$ is an example of a total order.

Given a set X, the *power set* $\mathcal{P}(X)$ is the set of all subsets of X. The subset relation $\subseteq$ on $\mathcal{P}(X)$ is a partial order.

- A relation R is *symmetric* if

$$aRb \Rightarrow bRa.$$

A reflexive, transitive, and symmetric relation is called an *equivalence relation*, denoted

$$a \sim b.$$

Let R be an equivalence relation on X. For each $a \in A$, we may define the subset

$$[a] := \{x \in X \,:\, x \sim a\},$$

called the *equivalence class* of a.

Check that if $b \sim a$, then $[b] = [a]$. Also, each a lies in exactly one equivalence class.

The collection of equivalence classes,

$$X/_\sim := \{[a] \,:\, a \in X\},$$

is the *quotient set* of X by the relation R.

Distinct but equivalent elements of X are "collapsed" to a single element in the quotient set, thereby losing their distinguishability.

- We have an intuitive concept of subtraction of natural numbers, denoted $m - n$, obeying the rule

$$m - n = m' - n' \quad \text{iff} \quad m + n' = m' + n.$$

To formalize this concept in terms of addition, let us consider ordered pairs (m, n) of natural numbers, but declare that

$$(m, n) \sim (m', n') \quad \text{iff} \quad m + n' = m' + n.$$

Check that $\sim$ is indeed an equivalence relation on $\mathbb{N} \times \mathbb{N}$. The quotient set is denoted

$$\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/_\sim,$$

and we define the addition operation on $\mathbb{Z}$ by

$$\begin{aligned} \mathbb{Z} \times \mathbb{Z} &\to \mathbb{Z} \\ \big([(m, n)], [(m', n')]\big) &\mapsto [(m, n)] + [(m', n')] := [(m + m'), (n + n')]. \end{aligned}$$

Check that this addition rule is well-defined, commutative, and associative.

  - There is an injective (check!) map

$$\begin{aligned} \iota : \mathbb{N} &\to \mathbb{Z} \\ n &\mapsto [(n, 0)], \end{aligned}$$

    thus a copy of $\mathbb{N}$ inside $\mathbb{Z}$.

    The addition on $\mathbb{Z}$ restricts to the addition on this copy of $\mathbb{N}$.

– The zero element $0 \in \mathbb{N}$, identified as $[(0,0)]$ inside $\mathbb{Z}$, is zero element of $\mathbb{Z}$, since

$$[(m,n)] + [(0,0)] = [(m,n)]$$

for every $[(m,n)] \in \mathbb{Z}$.

The additive inverse of $[(m,n)]$ is

$$-[(m,n)] = [(n,m)].$$

In particular,

$$-[(n,0)] = [(0,n)].$$

Just as $[(n,0)] \in \mathbb{Z}$ corresponds to $n \in \mathbb{N}$, it is conventional to denote $[(0,n)] \in \mathbb{Z}$ by $-n$. In this notation, for $m, n \in \mathbb{N}$, we have

$$m - n = [(m,0)] + [(0,n)] = [(m,n)].$$

– Every integer $[(m,n)]$ is equal to either $[(k,0)]$ or $[(0,k)]$ for some natural number k, and is said to be *non-negative* or *non-positive*, respectively. Accordingly, we just write

$$[(m,n)] = k \quad \text{or} \quad [(m,n)] = -k,$$

and so

$$\mathbb{Z} = \{\ldots, -(k+1), -k, \ldots, -1, 0, 1, \ldots, k, k+1, \ldots\}.$$

For negative integers $-k$, we write $-(-k) = k$. Then subtraction of integers is

$$a - b := a + (-b), \qquad a, b \in \mathbb{Z}.$$

- The above construction is called the *Grothendieck construction* applied to the abelian monoid $(\mathbb{N}, +)$.

- The ordering relation $\leq$ on $\mathbb{N} \times \mathbb{N}$ can be extended to $\mathbb{Z} \times \mathbb{Z}$:

$$a \leq b \quad \text{iff} \quad \exists k \in \mathbb{N} \text{ such that } a + k = b.$$

Check that this is still a total order with the trichotomy property, and addition of integers preserves the ordering.

- *Multiplication* on $\mathbb{N}$ is defined recursively by

$$\begin{aligned} \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \\ (0, n) &\mapsto 0 \times n := 0 \\ ((S(m), n)) &\mapsto S(m) \times n := (m \times n) + n, \end{aligned}$$

  and we often just write

$$mn \equiv m \times n.$$

  Using Peano axioms, we may verify that multiplication is commutative, associative, and

$$a(b + c) = ab + ac = ba + ca + (b + c)a. \qquad \text{(distributive law)}$$

  These properties are consistent with the ordinary usage of multiplication to reason about areas and volumes.

  We extend multiplication to integers by defining

$$\begin{aligned} \mathbb{Z} \times \mathbb{Z} &\to \mathbb{Z} \\ \big([(m, n)], [(m', n')]\big) &\mapsto [(m, n)] \times [(m', n')] := [(mm' + nn', mn' + nm')]. \end{aligned}$$

  Check well-definedness: For example, because $[(m, n)] = [(m+k, n+k)]$, we should have

$$\begin{aligned} [(mm' + nn', mn' + nm')] &\overset{\text{definition}}{=} [(m, n)] \times [(m', n')] \\ &\overset{\text{same equiv. class}}{=} [(m + k, n + k)] \times [(m', n')] \\ &\overset{\text{definition}}{=} \big[\big((m + k)m' + (n + k)n'), (m + k)n' + (n + k)m'\big)\big]. \end{aligned}$$

  The left side is indeed the same equivalence class as the right side (why?).

  In subtraction notation, the above multiplication law is the familiar high school formula

$$(m - n) \times (m' - n') = (mm' + nn') - (mn' + nm'). \qquad (m, n \in \mathbb{N})$$

  Also check that commutativity, associativity, distributivity laws continue to hold for addition/multiplication of integers.

  Observe that, in the simplified notation $k \in \mathbb{Z}$, we have

$$\begin{aligned} k \times 1 &= k, \\ k \times (-1) &= -k. \end{aligned}$$

- Exercise. Let $a, b \in \mathbb{Z}$ and $c \in \mathbb{Z} \setminus \{0\}$. Show that:

No zero divisors: If $ab = 0$, then $a = 0$ or $b = 0$.

Cancellation: If $ac = bc$, then $a = b$.

Exercise. Let $a, b, c \in \mathbb{Z}$. Show that:

- If $a < b$ and $0 < c$, then $ac < bc$.
- If $a < b$, $-b < -a$.

- A *sequence* of objects is generally denoted $x_1, x_2, \ldots, x_n, \ldots$. The subscript labels come from the *ordered* set of natural numbers, and they indicate whether x_m comes before x_n or vice versa.

  Natural numbers are also used to *count* the number of elements in a finite set X. Specifically, we say that X is *finite* with *cardinality* $|X| = n \in \mathbb{N}$, if there exists a bijection from X to the set

  $$\{k \in \mathbb{N} \,:\, 1 \leq k \leq n\}.$$

  Check that a finite set has a *unique* cardinality $n \in \mathbb{N}$.

  A general set X may not be finite in the above sense, then it is called an *infinite* set. It is still meaningful to compare the cardinalities $|X|, |Y|$ of general sets X, Y, by saying that

  $$|X| = |Y| \;\Leftrightarrow\; \exists \text{ bijection } f : X \to Y.$$

  Check that the relation of being in bijection is an equivalence relation (some set-theoretic technicalities here!) The various distinct cardinalities $|X|$ are called *cardinal* "numbers", and natural numbers may be viewed as the *finite* cardinal numbers.

  - We proved that $\mathbb{N}$ is not finite (i.e., it is an infinite set), by contradiction:
    - $\mathbb{N}$ is non-empty, so it does not have cardinality 0.
    - Assume that $|\mathbb{N}| = n$ for some $n \in \mathbb{N}$, i.e., there is a bijection $f : \mathbb{N} \to \{1, \ldots, n\}$.
    - Then f restricts to a bijection from $\mathbb{N}\setminus\{0\}$ to $\{1, \ldots, n\} \setminus \{f(0)\}$. The latter set has cardinality $n - 1$ (why?), so $|\mathbb{N}| = n - 1$.

* But there is also a bijection

$$\begin{aligned} \mathbb{N} &\to \mathbb{N} \setminus \{0\} \\ n &\mapsto n+1, \end{aligned}$$

thus

$$n = |\mathbb{N}| = |\mathbb{N} \setminus \{0\}| = n - 1.$$

* The above contradiction, $n = n-1$, means that our assumption $|\mathbb{N}| = n$ was false, for every $n \in \mathbb{N} \setminus \{0\}$. (Also false for $n = 0$.)

– The set $2\mathbb{N} := \{2n \,:\, n \in \mathbb{N}\}$ is an infinite subset of $\mathbb{N}$. Nevertheless, the map

$$\begin{aligned} f : \mathbb{N} &\to 2\mathbb{N} \\ n &\mapsto 2n \end{aligned}$$

is a bijection, so

$$|2\mathbb{N}| = |\mathbb{N}|.$$

Similarly,

$$|\mathbb{N}| = |\mathbb{Z}|.$$

- Cardinalities of *finite* sets behave like natural numbers:

$$\begin{aligned} |X \cup Y| &\leq |X| + |Y|, \qquad (= \text{ holds for disjoint union}), \\ |X \times Y| &= |X| \times |Y|, \\ |\mathcal{P}(X)| &= 2^{|X|}. \end{aligned}$$

Exponentiation of natural numbers is recursively defined by

$$\begin{aligned} \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \\ (m, 0) &\mapsto m^0 := 1, \\ (m, S(n)) &\mapsto m^{S(n)} := m^n \times m. \end{aligned}$$

Cantor discovered that *infinite* sets can have distinct cardinalities (see real numbers later). "Arithmetic" of infinite cardinal numbers is a tricky issue, and for now, we do not comment further on this issue.

- The multiplicative identity element is 1. But *not* every $k \in \mathbb{Z}$ has a multiplicative inverse $k^{-1} \in \mathbb{Z}$ satisfying

$$kk^{-1} = 1 = k^{-1}k.$$

This causes a problem in *coordinate geometry* — a system of labelling geometric points using numbers. Intuitively, a line segment I can be doubled, tripled, etc., in length. We should also be able to consider a shorter line segment J of length "k^{-1}" such that k times of J is I. To describe such "divided" intervals with numbers, we have to introduce *rational* numbers, constructed, e.g., below.

- Take ordered pair $(p, q) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, denoted $\frac{p}{q}$ by convention. Declare

$$\frac{p}{q} \sim \frac{p'}{q'} \quad \text{iff} \quad pq' = p'q,$$

and check that this is an equivalence relation. The set of rational numbers is the quotient set

$$\mathbb{Q} := \big(\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})\big)/_{\sim}.$$

By convention, we just write $\frac{p}{q} \in \mathbb{Q}$ instead of using equivalence class notation $\left[\frac{p}{q}\right] \in \mathbb{Q}$.

Integers are identified as a subset of rational numbers, by the injective map

$$k \mapsto \frac{k}{1},$$

and we just write $k \in \mathbb{Q}$ for convenience.

- Addition and multiplication of rational numbers are well-defined operations,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$
$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd},$$

extending the addition and multiplication of integers. Check that they remain commutative, associative, and that distributive law still holds.

The additive identity is 0, and the additive inverse of $\frac{a}{b}$ is

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}.$$

- The extra property that $\mathbb{Q}$ enjoys is the existence of *multiplicative inverse* for the *nonzero* elements. Explicitly,

$$\left(\frac{p}{q}\right)^{-1} = \frac{q}{p}. \qquad (p, q \neq 0).$$

The inverse $\frac{q}{p}$ is called the *reciprocal* of $\frac{p}{q}$.

Then, just as subtraction means addition of additive inverse, *division* means multiplication of reciprocal,

$$x \div y := x \times y^{-1}, \qquad x \in \mathbb{Q}, y \in \mathbb{Q} \setminus \{0\}.$$

- A nonzero $x \in \mathbb{Q}$ is

$$\begin{cases} \text{positive}, \\ \text{negative}, \end{cases} \quad \text{iff} \quad \exists a, b \in \mathbb{N} \setminus \{0\} \text{ such that } x = \begin{cases} \frac{a}{b}, \\ -\frac{a}{b}, \end{cases}$$

with exactly one case being true (why?).

The ordering $\leq$ on $\mathbb{Z}$ extends to a total order on $\mathbb{Q}$,

$$x < y \text{ if } y - x \text{ positive},$$
$$x \leq y \text{ if } x < y \text{ or } x = y.$$

Exercise: Check that the trichotomy property still holds, that addition preserves $\leq$, and that multiplication by positive $c \in \mathbb{Q}$ preserves $<$. In particular,

$$0 \leq x \text{ and } 0 \leq y \;\Rightarrow\; 0 \leq xy.$$

In brief, one says that $\mathbb{Q}$, with $+, \times, \leq$, is an *ordered field*. Whereas $\mathbb{Z}$ is only an *ordered ring*.

- Cantor's diagonal argument shows that

$$|\mathbb{Q}| = |\mathbb{Z}| = |\mathbb{N}|.$$

(But the bijection $\mathbb{N} \to \mathbb{Q}$ does not give a sequence of *increasing* rational numbers.)