9 Lectures 17–18

- We will now use (Corollary 8.6 of) the MVT, and the power series derivative formula (Prop. 8.2) to prove some basic trigonometric derivatives and identities.

  – Let $f(x) = \sin^2(x) + \cos^2(x)$. By the product rule and linearity of derivatives, and (8.19),

  $$\begin{aligned} f'(x) &= 2\sin(x)\sin'(x) + 2\cos(x)(\cos'(x)) \\ &= 2(\sin(x)\cos(x) - \cos(x)\sin(x)) = 0, \qquad x \in \mathbb{R}. \end{aligned}$$

  By Corollary 8.6, f is constant on $\mathbb{R}$. Since $f(0) = 0^2 + 1^2 = 1$, we deduce the trigonometric Pythagoras identity,

  $$\sin^2(x) + \cos^2(x) = 1. \tag{9.1}$$

  This implies that $\sin, \cos$ are bounded functions.

  * Recall that
  $$\cos\big(\frac{\pi}{2}\big) = 0. \tag{9.2}$$
  More precisely, we had defined $\frac{\pi}{2}$ to be the smallest positive x such that $\cos(x) = 0$. We must have also have
  $$\cos(x) > 0, \qquad x \in [0, \tfrac{\pi}{2}), \tag{9.3}$$
  otherwise, $\cos(0) = 1$ and IVT would lead to an even smaller positive solution to $\cos(x) = 0$ (work out the details).
  * By (9.1), we know that $\sin(\frac{\pi}{2}) = \pm 1$. By (9.3), (8.19), and Exercise 8.5, sin is strictly increasing on $[0, \frac{\pi}{2}]$. Thus
  $$\sin\big(\frac{\pi}{2}\big) = +1. \tag{9.4}$$

  – Similarly, use Corollary 8.6 to prove the following sine/cosine addition identities.

  Exercise 9.1. Show that

  $$\begin{aligned} \sin(x \pm y) &= \sin(x)\cos(y) \pm \cos(x)\sin(y), \\ \cos(x \pm y) &= \cos(x)\cos(y) \mp \sin(x)\sin(y), \qquad x, y \in \mathbb{R}. \end{aligned}$$

Since $\sin(0) = 0$ and $\cos(0) = 1$, Exercise 9.1 shows that

$$\sin(-y) = -\sin(y), \qquad \cos(-y) = \cos(y), \qquad y \in \mathbb{R}. \tag{9.5}$$

(The odd/even properties in (9.5) is also immediate from the power series definitions.)

Combining (9.5) with Exercise 8.5, (8.19), and (9.3), it follows easily that sin is strictly monotone increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, while cos is strictly monotone decreasing on $[0, \pi]$. We can therefore invert sin and cos on these domains, to get

$$\begin{aligned} \arcsin \equiv \sin^{-1} &: [-1, 1] \to [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \subseteq \mathbb{R}, \\ \arccos \equiv \cos^{-1} &: [-1, 1] \to [0, \pi] \subseteq \mathbb{R}, \end{aligned} \tag{9.6}$$

which are continuous by Theorem 7.4. Furthermore, we can use the inverse function derivative formula (8.12) from Exercise 8.3 to deduce:

Exercise 9.2. Show that

$$\begin{aligned} \arcsin'(x) &= \frac{1}{\sqrt{1 - x^2}}, \\ \arccos'(x) &= -\frac{1}{\sqrt{1 - x^2}}, \qquad x \in (-1, 1). \end{aligned} \tag{9.7}$$

– By Exercise 9.1 and (9.2)–(9.4), we get

$$\sin(x + \tfrac{\pi}{2}) = \cos(x), \qquad \cos(x + \tfrac{\pi}{2}) = -\sin(x), \qquad x \in \mathbb{R}. \tag{9.8}$$

Thus,

$$\sin(x+2\pi) = \cos(x+\tfrac{3\pi}{2}) = -\sin(x+\pi) = -\cos(x+\tfrac{\pi}{2}) = \sin(x), \qquad x \in \mathbb{R}, \tag{9.9}$$

and similarly, $\cos(x + 2\pi) = \cos(x)$. Thus the sine and cosine functions are 2π-periodic functions.

– From the above discussion, we deduce that the set of zeroes for sin is $n\pi, n \in \mathbb{Z}$, while the set of zeroes for cos is $\frac{\pi}{2} + n\pi, n \in \mathbb{Z}$. Away from the respective zero sets, we can define

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \qquad \cot(x) = \frac{\cos(x)}{\sin(x)}.$$

From (9.8) and (9.5), we also obtain

$$\tan(\tfrac{\pi}{2} - x) = \frac{\cos(-x)}{-\sin(-x)} = \frac{\cos(x)}{\sin(x)} = \cot(x), \qquad x \in (0, \tfrac{\pi}{2}).$$

Goemetric interpretation: For a right-angled triangle in the Euclidean plane, with angles $\frac{\pi}{2}, x, \frac{\pi}{2} - x$, the cotangent of the angle x is the tangent of the complementary angle $\frac{\pi}{2} - x$.

Their derivatives are easily computed, using the quotient rule and (9.1), to be

$$\begin{aligned} \tan'(x) &= \frac{1}{\cos^2(x)} = 1 + \tan^2(x), \\ \cot'(x) &= -\frac{1}{\sin^2(x)} = -(1 + \cot^2(x)). \end{aligned} \tag{9.10}$$

Because of (9.10), the functions

$$\begin{aligned} \tan &: (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \to \mathbb{R} \\ \cot &: (0, \pi) \to \mathbb{R} \end{aligned}$$

are strictly monotonic, thus injective. Their ranges are $\mathbb{R}$. (Exercise: check that $\tan, \cot$ are unbounded above and below, then use IVT to deduce that all real numbers are attained in their ranges.) So we can invert them, and by formula (8.12) and (9.10), the derivatives of their inverse functions are

$$\begin{aligned} \arctan'(x) &= \frac{1}{1 + x^2}, \\ \operatorname{arccot}'(x) &= -\frac{1}{1 + x^2}, \qquad\qquad x \in \mathbb{R}. \end{aligned}$$

Exercise 9.3. Consider the following functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$,

$$f_1 : x \mapsto \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad f_2 : x \mapsto \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

Determine whether f_1, f_2 are continuous and/or differentiable at $x = 0$.

- Suppose $x \equiv x(t)$ is *continuously* differentiable on an open interval, i.e., x' is a continuous function on I. Then, on any subinterval $[a, b] \subseteq I$,

we may apply the IVT to x'. Interpret x' as the velocity function, $x'(a)$ as the initial velocity, and $x'(b)$ as the final velocity. The conclusion from the IVT is that every velocity in between $x'(a)$ and $x'(b)$ must be attained at some time during the time interval (a, b). This fact is clear from observing a good car driver.

Remarkably, this kind of "Intermediate Value Theorem" for derivatives does not require the assumption of continuity of the derivative function[20]. The IVT applies to the velocity function, even for a poor driver!

Lemma 9.1. *If f has positive left-sided derivative at x_0, then there exists $\delta > 0$ such that $f(x) < f(x_0)$ for all $x \in (x_0 - \delta, x_0)$. If f has negative right-sided derivative at x_0, then there exists $\delta > 0$ such that $f(x) < f(x_0)$ for all $x \in (x_0, x_0 + \delta)$.*

Proof. Exercise. □

Theorem 9.2 (Darboux's Theorem)**.** *Let f be a real-valued function which is differentiable on an open interval I. For any $a, b \in I$ with $a < b$, and any λ strictly between $f'(a)$ and $f'(b)$, there exists $t \in (a, b)$ such that $f'(t) = \lambda$.*

Proof. It suffices to consider the case $f'(a) < f'(b)$. Given $\lambda \in \big(f'(a), f'(b)\big)$, consider the function

$$g(t) = f(t) - \lambda t, \qquad t \in [a, b].$$

Then $g'(t) = f'(t) - \lambda$, so

$$g'(a) < 0, \qquad g'(b) > 0.$$

By Lemma 9.1, there exists $\delta > 0$ such that

$$t \in (a, a + \delta) \Rightarrow g(t) < g(a),$$
$$t \in (b - \delta, b) \Rightarrow g(t) < g(b).$$

Thus, neither a nor b is a global minimum for g over the interval $[a, b]$. At the same time, g is continuous on $[a, b]$, so by the Extreme Value

[20]The key lesson is that a derivative function is a special kind of function — not all functions on an interval can arise as a derivative function.

Theorem, it attains its minimum at some $t \in (a, b)$. Since t is also a local minumum for g, by Fermat's Stationary Point Theorem (Lemma 8.3),

$$0 = g'(t) = f'(t) - \lambda.$$

□

Example 9.1. A classic example of a differentiable, but not *continuously* differentiable function, is the function introduced in Exercise 9.3,

$$f_2 : t \mapsto \begin{cases} t^2 \sin(1/t), & t \neq 0, \\ 0, & t = 0. \end{cases}$$

The derivatives at $t \neq 0$ are easily obtained via chain rule and product rule,

$$f_2'(t) = 2t \sin \frac{1}{t} - \cos \frac{1}{t}, \qquad t \neq 0,$$

while $f_2'(0) = 0$ should have been calculated in Exercise 9.3. Now, by 2π-periodicity and $\sin(0) = 0 = \sin \pi$ and $\cos(0) = 1$, we easily obtain

$$f_2' \left(\frac{1}{2\pi n} \right) = -1, \qquad n \geq 1.$$

So setting $x_n = \frac{1}{2\pi n}$, we get a sequence converging to 0, but

$$\lim_{n \to \infty} f_2'(x_n) = -1 \neq f_2'(0),$$

i.e., f_2' is not continuous at 0.

Exercise 9.4. Suppose $x \equiv x(t)$ is differentiable on an open interval I, with $x'(t) \neq 0$ for all $t \in I$. Deduce from Darboux's Theorem (and Exercise 8.5) that x is strictly monotone on I. Then use other results proved in earlier lectures to show that $x(I)$ is an open interval, and that the inverse function $x^{-1} : x(I) \to I \subseteq \mathbb{R}$ is differentiable on $x(I)$.

- Let $x \equiv x(t)$ and $y \equiv y(t)$ be differentiable functions of a *parameter* $t \in I$, where I is an open interval. Suppose $x'(t) \neq 0$ for all $t \in I$, then by Exercise 9.4, the inverse function $x^{-1} : x(t) \mapsto t$ exists and is differentiable on the interval $x(I)$. So, instead of regarding y as a function of x, we can regard y as a function of t; more precisely, we write

$$y = \tilde{y} \circ x,$$

where

$$\tilde{y} := y \circ x^{-1} : x(I) \to \mathbb{R}$$
$$x(t) \mapsto y(t).$$

Then the chain rule and (8.12) give

$$\tilde{y}'(x(t)) = (y \circ x^{-1})'(x(t)) = y'(t) \cdot (x^{-1})'(x(t)) = \frac{y'(t)}{x'(t)}, \qquad t \in I. \tag{9.11}$$

If $y'(t) \neq 0$ for all $t \in I$, we swap the roles of x, y to get

$$\tilde{x}'(y(t)) = \frac{x'(t)}{y'(t)}, \qquad t \in I. \tag{9.12}$$

The formulae (9.11)–(9.12) are often written informally as

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \qquad \frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}}, \tag{9.13}$$

with $y \equiv y(x)$ on the left side of the equality and $y \equiv y(t)$ on the right side.

Let us explain the geometric meaning of (9.11)–(9.12) (or (9.13)). The map

$$\gamma : t \mapsto \big(x(t), y(t)\big) \tag{9.14}$$

defines a (differentiable) *parametrized curve* in the plane $\mathbb{R}^2$. Its "velocity" at time $t \in I$ is $\gamma'(t) := \big(x'(t), y'(t)\big)$. When $\gamma'(t) \neq (0,0)$, we can use one the formulas (9.11)–(9.12) to get the (non-zero) slopes/tangents of the function $\tilde{y} \equiv \tilde{y}(x)$ and the function $\tilde{x} \equiv \tilde{x}(y)$ at the point $(x(t), y(t))$ along the curve γ. They are reciprocals of one another.

Example 9.2. For the functions

$$x(t) = \cos(t), \qquad y(t) = \sin(t), \qquad t \in \mathbb{R},$$

the curve $\gamma(t) = \big(x(t), y(t)\big)$ parametrizes the unit circle $S^1 \subseteq \mathbb{R}^2$. (The Pythagoras identity leads to the level set description $x^2 + y^2 = 1$. To be injective, sometimes one restricts the parametrization interval to, e.g., $(-\pi, \pi]$.)

Restricting to $t \in (0, \pi)$ or $t \in (-\pi, 0)$, the formula (9.11) is

$$\frac{d\tilde{y}}{dx}\Big|_{x(t)} = \frac{y'(t)}{x'(t)} = \frac{\cos(t)}{-\sin(t)} = -\frac{x(t)}{y(t)}, \qquad t \in (-\pi, 0) \cup (0, \pi). \tag{9.15}$$

If you are familiar with the standard dot product (inner product) in $\mathbb{R}^2$), we have the *orthogonality* relation

$$\gamma(t) \cdot \gamma'(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \cdot \begin{pmatrix} -y(t) \\ x(t) \end{pmatrix} = -x(t)y(t) + y(t)x(t) = 0,$$

between the position and the velocity vectors, at all times during the circular motion.

To describe general plane curves which avoid $(0, 0)$, it is often convenient to use an "angular" parameter θ, and a positive, differentiable "radial" function $r \equiv r(\theta)$. In Cartesian coordinates, we would then have

$$\begin{aligned} x(\theta) &= r(\theta) \cos \theta, \\ y(\theta) &= r(\theta) \sin \theta, \qquad \theta \in I. \end{aligned}$$

The parametrized circle in Example 9.2 is a special case, with r being the constant function 1.

Let I be an open parametrization interval on which $x'(\theta) \neq 0$. Then formula (9.11) and the Leibniz rule give

$$\frac{d\tilde{y}}{dx}\Big|_{x(\theta)} = \frac{r'(\theta) \sin \theta + r(\theta) \cos \theta}{r'(\theta) \cos \theta - r(\theta) \sin \theta}. \tag{9.16}$$

Example 9.3. For $a \neq 0$, the *Archimedes spiral* is the plane curve defined by $r(\theta) = a\theta$, $\theta \in (0, \infty)$. We could try to locally solve for its y-coordinate function in terms of its x-coordinate function, then take the derivative $\frac{dy}{dx}$. But it is easier to use the formula (9.16) to get

$$\frac{d\tilde{y}}{dx}\Big|_{x(\theta)} = \frac{\sin \theta + \theta \cos \theta}{\cos \theta - \theta \sin \theta} = \frac{y(\theta) + \theta \cdot x(\theta)}{x(\theta) - \theta \cdot y(\theta)}, \qquad \theta \neq \cot \theta. \tag{9.17}$$

While (9.17) is not an explicit formula in terms of the x-coordinate, it provides some useful information, e.g., the slope of an Archimedes spiral's tangent depends only on the angular parameter θ, and not on a.

- There is another version of MVT, which will be important for the proof of the L'Hôpital rule in a later lecture.

  Theorem 9.3 (Cauchy's Mean Value Theorem)**.** *Let $x, y : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). There exists $t \in (a, b)$ such that*

  $$\big(x(b) - x(a)\big)y'(t) = \big(y(b) - y(a)\big)x'(t). \tag{9.18}$$

  Proof. Define the function $f : [a, b] \to \mathbb{R}$ by

  $$f = \big(x(b) - x(a)\big)\big(y - y(a)\big) - \big(y(b) - y(a)\big)\big(x - x(a)\big).$$

  Then f is continuous on $[a, b]$ and differentiable on (a, b), and it satisfies $f(a) = 0 = f(b)$. By Rolle's Theorem, there exists $t \in (a, b)$ such that $f'(t) = 0$, equivalently, (9.18) holds. □

  Corollary 9.4 (Cauchy's MVT)**.** *Let $x, y : I \to \mathbb{R}$ be differentiable on some open interval I. Assume that $x'(t) \neq 0$ for all $t \in I$. Then for any $a, b \in I$ with $a < b$, there exists $t \in (a, b)$ such that*

  $$\frac{y(b) - y(a)}{x(b) - x(a)} = \frac{y'(t)}{x'(t)}. \tag{9.19}$$

  Proof. The Cauchy MVT applied to $x, y : [a, b] \to \mathbb{R}$ gives $t \in (a, b)$ satisfying (9.18). As $x'(t) \neq 0$ for all $t \in (a, b)$, the contrapositive of Rolle's Theorem says that $x(a) \neq x(b)$. So we can divide (9.18) by the nonzero number $(x(b) - x(a)) \cdot x'(t)$ to get (9.19). □

  – The left side of (9.19) is the slope of the straight line ℓ_{ab} joining $(x(a), y(a))$ to $(x(b), y(b))$. The right side of (9.19) is just the parametric derivative formula, $\frac{d\tilde{y}}{dx}\Big|_{x(t)}$, i.e., the slope of the tangent to the curve $\gamma : t \mapsto (x(t), y(t))$ at the time parameter t. Just like in Rolle's Theorem or Lagrange's MVT, the intuitive idea is: "parallel transport" the line ℓ_{ab} "outwards" until it just "leaves" the curve γ. The point of departure occurs at some intermediate $t \in (a, b)$, and the transported line is tangent to the curve there. Cauchy's MVT is the rigorous formulation of this idea.

– In Cauchy's MVT, the special case $x(t) = t$ is Lagrange's MVT. As mentioned previously, Lagrange's MVT is equivalent to Rolle's Theorem. Since we used Rolle's Theorem to prove Cauchy's MVT, we see that Cauchy's MVT is also equivalent to Lagrange's MVT.

- It is sometimes useful to say that a function like $x \mapsto \frac{1}{x}$ "tends to $+\infty$" as $x \to 0^+$".

  Definition 9. Let $E \subseteq \mathbb{R}$, x_0 be a limit point of E, and $f : E \setminus \{x_0\} \to \mathbb{R}$. We say that f tends to $\begin{cases} +\infty \\ -\infty \end{cases}$ at x_0, if

$$\forall \mu \in \mathbb{R},\, \exists \delta > 0,\, \forall x \in E, \quad 0 < |x - x_0| < \delta \;\Rightarrow\; f(x) \begin{cases} > \mu, \\ < \mu \end{cases} .$$

  Remark. As with ordinary limits,

  – If $E \setminus \{x_0\}$ is a deleted neighbourhood of x_0, then we write

$$\lim_{x \to x_0} f(x) = \pm\infty,$$

    when f tends to $\pm\infty$ at x_0.

  – If $E \setminus \{x_0\} = (x_0, x_0 + \delta_0)$ (respectively, $(x_0 - \delta_0, x_0)$) for some $\delta_0 > 0$, we are in the right-sided (respectively, left-sided) limit situation. In this case, we write

$$\lim_{x \to x_0^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to x_0^-} f(x) = \pm\infty$$

    when f tends to $\pm\infty$ at x_0, respectively.

  – For general E, we should write $\lim_{x \to x_0; x \in E \setminus \{x_0\}}$ for clarity, but we rarely have to deal with such situations.

  It is also useful to say that $x \mapsto \frac{1}{x}$ has "left-sided limit" 0 as x "tends to $+\infty$", and has "right-sided limit" 0 as x "tends to $-\infty$".

  Definition 10. Let f be a real-valued function defined on an unbounded interval I. If I is unbounded above (respectively, unbounded below), we say that $L \in \mathbb{R}$ is the *limit of f at $+\infty$* (respectively, at $-\infty$), if

$$\forall \epsilon > 0,\, \exists M \in I, \forall x \in I, \quad \begin{cases} x > M \\ x < M \end{cases} \quad \Rightarrow\; |f(x) - L| < \epsilon.$$

In this case, we write

$$\begin{cases} \lim\limits_{x\to+\infty} f(x) = L, \\ \lim\limits_{x\to-\infty} f(x) = L. \end{cases}$$

Remark. There are some obvious generalizations of Definition 10.

- When $M \equiv M(\epsilon)$ works, any larger $M' > M$ also works.
- We could replace I by some unbounded subset $E \subseteq \mathbb{R}$.
- We can combine Definitions 9 and 10, and talk about a function which tends to $\pm\infty$ at $\pm\infty$. For example, the function $x \mapsto \sinh x$ tends to $+\infty$ at $+\infty$, and tends to $-\infty$ at $-\infty$.

Exercise 9.5. Let $a > 0$ and $f : (a, +\infty) \to \mathbb{R}$. Define

$$\begin{aligned} \tilde{f} : (0, a^{-1}) &\to \mathbb{R} \\ t &\mapsto f(t^{-1}). \end{aligned}$$

Show that the "left-sided limit" $\lim\limits_{x\to+\infty} f(x)$ exists (as an extended real number) iff the right-sided limit $\lim\limits_{t\to0^+} \tilde{f}(t)$ exists, in which case,

$$\lim_{x\to+\infty} f(x) = \lim_{t\to0^+} \tilde{f}(t).$$

Write down the analogous version of this Exercise for $g : (-\infty, -a) \to \mathbb{R}$ and $\lim\limits_{t\to0^-} \tilde{g}(t)$.

Definition 11. A sequence $x_1, x_2 \ldots$ of real numbers *tends to* $+\infty$ (respectively, $-\infty$) if

$$\forall \mu \in \mathbb{R},\ \exists N \geq 1,\ \forall n \geq N, \quad x_n > \mu \text{ (respectively, } x_n < \mu).$$

In this case, we write

$$\lim_{n\to\infty} x_n = +\infty \quad \text{(respectively, } \lim_{n\to\infty} x_n = -\infty).$$

Exercise 9.6. Show that Definition 9 has an equivalent sequential version,

$$\forall \text{ sequences } x_1, x_2, \ldots \text{ in } E,\ \lim_{n\to\infty} x_n = x_0 \ \Rightarrow\ \lim_{n\to\infty} f(x_n) = \begin{cases} +\infty, \\ -\infty. \end{cases}$$

Show that Definition 10 has an equivalent sequential version,

$$\forall \text{ sequences } x_1, x_2, \dots \text{ in } I, \begin{cases} \lim_{n\to\infty} x_n = +\infty \\ \lim_{n\to\infty} x_n = -\infty \end{cases} \Rightarrow \lim_{n\to\infty} f(x_n) = L.$$

- Suppose f, g have limits L, M at x_0, respectively. In Exercise 5.6, we saw that if $M \neq 0$, then $\lim_{x\to x_0} \frac{f}{g}(x) = \frac{L}{M}$ exists.

  What happens if $M = 0$? Suppose $\lim_{x\to x_0} \frac{f}{g}(x) = K \in \mathbb{R}$ exists. Then we must have

  $$L = \lim_{x\to x_0} f(x) = \lim_{x\to x_0} \frac{f}{g}(x) \cdot \lim_{x\to x_0} g(x) = K \cdot M = 0.$$

  This tells us that the case $L = 0 = M$ *might* allow for $\lim_{x\to x_0} \frac{f}{g}(x)$ to exist. The formula

  $$\lim_{x\to x_0} \frac{f}{g}(x) = \text{“ } \frac{L}{M} = \frac{0}{0} \text{ ”}$$

  is of course nonsensical. So we ask: Under what assumptions does $\lim_{x\to x_0} \frac{f}{g}(x)$ exist, and what replaces “$\frac{L}{M}$” as the correct formula for this limit?

Proposition 9.5 (Baby L'Hôpital's Rule)**.** *Let f, g be right-differentiable at $a \in \mathbb{R}$, with right-sided derivatives denoted $f'(a), g'(a)$ respectively. If*

- *$f(a) = g(a) = 0$;*
- *$g'(a) \neq 0$;*
- *$g(x) \neq 0$ on some interval (a, b);*

then

$$\lim_{x\to a^+} \frac{f}{g}(x) = \frac{f'(a)}{g'(a)}. \tag{9.20}$$

Proof. As $f(a) = 0 = g(a)$, and $g(x) \neq 0 = g(a)$, we have the identity

$$\frac{f}{g}(x) = \frac{(f(x) - f(a))/(x - a)}{(g(x) - g(a))/(x - a)}, \qquad \forall x \in (a, b).$$

By definition, the $x \to a^+$ limits of the numerator and denominator are the right-sided derivatives $f'(a)$ and $g'(a) \neq 0$, respectively. So, by the quotient rule for (one-sided) limits, formula (9.20) is valid. □

Proposition 9.5 refers to some differentiability assumptions, and uses only the definition of (right-sided) derivative. It does not use the main extra result for derivatives (MVT). Furthermore, it does provide a satisfactory answer to our question, because it uses function values *at* a itself. Whereas $\lim_{x\to a} \frac{f}{g}(x)$ is generally a quantity that does not depend on the data *at* a.

Theorem 9.6 (L'Hôpital's Rule, $\frac{0}{0}$ version)**.** *Let $a \in \{-\infty\} \sqcup \mathbb{R}$ and f, g be real-valued functions such that*[21]

1. $\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x)$*;*
2. *f, g are differentiable on some (a, b), with $g'(x) \neq 0$ for all $x \in (a, b)$;*
3. $\lim_{x\to a^+} \frac{f'}{g'}(x)$ *exists in $\{-\infty\} \sqcup \mathbb{R} \sqcup \{+\infty\}$.*

Then

$$\lim_{x\to a^+} \frac{f}{g}(x) = \lim_{x\to a^+} \frac{f'}{g'}(x). \tag{9.21}$$

Similarly, for $b \in \mathbb{R} \sqcup \{+\infty\}$,

$$\lim_{x\to b^-} \frac{f}{g}(x) = \lim_{x\to b^-} \frac{f'}{g'}(x) \tag{9.22}$$

holds when f, g satisfy Assumptions 1,2,3 with $\lim_{x\to b^-}$ instead of $\lim_{x\to a^+}$.

Proof. – We handle the $a \in \mathbb{R}$ case first.

∗ [Assumption 3, case $\lim_{x\to a^+} \frac{f'}{g'}(x) = L \in \mathbb{R}$.] Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$L - \epsilon < \frac{f'}{g'}(c) < L + \epsilon, \qquad \forall c \in (a, a + \delta) \subseteq (a, b). \tag{9.23}$$

Fix any $x \in (a, a+\delta)$. Use Assumption 1 to extend f, g continuously to $[a, x]$ by setting $f(a) = 0 = g(a)$. Then by Assumption 2, the conditions for (the Corollary to) Cauchy's MVT are satisfied by $f|_{[a,x]}, g|_{[a,x]}$, so we obtain $\gamma \in (a, x)$ such that

$$\frac{f'(\gamma)}{g'(\gamma)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}. \tag{9.24}$$

[21] When $a = -\infty$, $\lim_{x\to a^+}$ means $\lim_{x\to-\infty}$. When $b = +\infty$, $\lim_{x\to b^-}$ means $\lim_{x\to+\infty}$.

(Note that $g(x) \neq g(a) = 0$ holds as part of the proof of Cauchy's MVT.) We can set $c = \gamma$ in (9.23), and combine it with (9.24) to get

$$L - \epsilon < \frac{f(x)}{g(x)} < L + \epsilon, \qquad \forall x \in (a, a+\delta).$$

As $\epsilon > 0$ was arbitrary, we have shown that $\lim\limits_{x\to a^+} \frac{f}{g} = L$.

* [Assumption 3, case $\lim\limits_{x\to a^+} \frac{f'}{g'}(x) = +\infty$.] Given $M \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\frac{f'}{g'}(x) > M, \qquad \forall x \in (a, a+\delta). \tag{9.25}$$

For each $x \in (a, a+\delta)$, the Cauchy MVT gives $\gamma \in (a, x)$ such that (9.24) holds, then combining with (9.25), we get

$$\frac{f(x)}{g(x)} > M, \qquad x \in (a, a+\delta).$$

As $M \in \mathbb{R}$ was arbitrary, we have shown that $\lim\limits_{x\to a^+} \frac{f}{g}(x) = +\infty$.

* [Assumption 3, case $\lim\limits_{x\to a^+} \frac{f'}{g'}(x) = -\infty$.] Similar to the previous case.

– The "left-sided", $b \in \mathbb{R}$ case is proved in the same way.

– For the $a = -\infty$ case, Assumptions 1,2 say that f is defined on $(-\infty, b)$ for some $b < 0$. Define $\tilde{f} : (b^{-1}, 0) \to \mathbb{R}$ by $\tilde{f}(t) = f(t^{-1})$; similarly for $\tilde{g}$. Assumption 1 and Exercise 9.5 say that

$$\lim_{t\to 0^-} \tilde{f}(t) = \lim_{x\to -\infty} f(x) = 0 = \lim_{x\to -\infty} g(x) = \lim_{t\to 0^-} \tilde{g}(t).$$

Assumption 2 and the Chain Rule say that $\tilde{f}, \tilde{g}$ are differentiable on $(-b^{-1}, 0)$, and that

$$\tilde{g}'(t) = g'(t^{-1}) \cdot \frac{d(t^{-1})}{dt}\Big|_t = -\frac{g'(t^{-1})}{t^2} \neq 0 \qquad t \in (b^{-1}, 0).$$

Assumption 3, the Chain Rule, and Exercise 9.5 say that

$$\lim_{t\to 0^-} \frac{\tilde{f}'}{\tilde{g}'}(t) = \lim_{t\to 0^-} \frac{-f'(t^{-1})/t^2}{-g'(t^{-1})/t^2} = \lim_{t\to 0^-} \frac{f'}{g'}(t^{-1}) = \lim_{t\to 0^-} \widetilde{f'/g'}(t) = \lim_{x\to -\infty} \frac{f'}{g'}(x).$$

exists in the extended real numbers.

So $\tilde{f}, \tilde{g}$ satisfy the "left-sided" version of the Assumptions 1–3 with finite $b = 0 \in \mathbb{R}$. We had already verified L'Hôpital's rule in that setting, so we know that

$$\lim_{t \to 0^-} \frac{\tilde{f}}{\tilde{g}}(t) = \lim_{t \to 0^-} \frac{\tilde{f}'}{\tilde{g}'}(t).$$

Then by Exercise 9.5 again,

$$\lim_{x \to -\infty} \frac{f}{g}(x) = \lim_{x \to -\infty} \frac{f'}{g'}(x).$$

– The "right-sided", $b = +\infty$, case is similarly proved.

□