10 Lectures 19–20

- One useful way to visualize the right-sided L'Hôpital's Rule is to use y, x instead of f, g, and consider parametrized plane curves

$$\gamma : t \mapsto \big(x(t), y(t)\big), \qquad t \in (a, b).$$

  - Assumption 1 corresponds to γ approaching the origin $(0, 0)$ as $t \to a^+$.
  - Assumption 2 corresponds to differentiability of the curve, and $x'(t) \neq 0$ for $t \in (a, b)$.
  - Assumption 3 corresponds to the existence of the $t \to a^+$ limit of

$$\frac{y'}{x'}(t) \overset{(9.11)}{=} \frac{d\tilde{y}}{dx}\Big|_{x(t)}.$$

    So the slope of the curves' tangents tend towards some "initial slope" as $t \to a^+$.

  Then L'Hôpital's Rule says that $\frac{y(t)}{x(t)}$ tends this "initial slope". This is clear from the intuitive picture illustrating Cauchy's MVT.

  As a special case, let us consider an angle-parametrized curve,

$$x(\theta) = r(\theta)\cos\theta, \qquad y(\theta) = r(\theta)\sin\theta, \qquad \theta \in (a, b).$$

  The L'Hôpital's Rule assumptions become

$$\lim_{\theta \to a^+} r(\theta) = 0, \tag{10.1}$$

  differentiability of r on (a, b), with

$$x'(\theta) = r'(\theta)\cos\theta - r(\theta)\sin\theta \neq 0, \qquad \theta \in (a, b), \tag{10.2}$$

  and

$$\lim_{\theta \to a^+} \frac{y'}{x'}(\theta) \overset{(9.16)}{=} \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta} \quad \text{exists.} \tag{10.3}$$

  (Here, we used knowledge of $\cos' = -\sin$ and $\sin' = \cos$.)

  If we make a further mild assumption that $\lim\limits_{\theta \to a^+} r'(\theta)$ exists, then we can substitute (10.1) into (10.3) to get

$$\lim_{\theta \to a^+} \frac{y'}{x'}(\theta) = \lim_{\theta \to a^+} \tan\theta = \tan a.$$

So the conclusion of L'Hôpital's Rule is just a tautology,

$$\tan a = \lim_{\theta \to a^+} \frac{r(\theta)\sin\theta}{r(\theta)\cos\theta} = \lim_{\theta \to a^+} \frac{y}{x}(\theta) = \lim_{\theta \to a^+} \frac{y'}{x'}(\theta) = \tan a.$$

- If

  1. $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$;
  2. f, g are differentiable on some deleted neighbourhood V of $a \in \mathbb{R}$, with g' being nonzero on V;
  3. The *two-sided* limit $\lim_{x \to a} \frac{f'}{g'}(x)$ exists (in extended real numbers);

  then the left-sided and right-sided L'Hôpital's Rules are both valid at a, and equal. So we obtain the *two-sided* L'Hôpital's Rule:

$$\lim_{x \to a} \frac{f}{g}(x) = \lim_{x \to a} \frac{f'}{g'}(x). \tag{10.4}$$

- After learning about L'Hôpital's Rule, many students become eager to use it at any opportunity. It is a *powerful* tool in the sense of *efficiency* in establishing and computing *certain* limits with the help of *known derivatives*.

  However, the L'Hôpital's Rule is a *dangerous* rule.

  – The first common mistake is applying the rule even when Assumption 1 (indeterminate form) does not hold, e.g.,

$$\lim_{x \to 0} \frac{x+1}{3x+2} \overset{?}{=} \lim_{x \to 0} \frac{1}{3} = \frac{1}{2}.$$

    Fortunately, this is easy to avoid.

  – A second mistake is thinking that a "$\frac{0}{0}$" limit does not exist because L'Hôpital's Rule failed to work. For example,

$$\text{“}\frac{0}{0}\text{”} = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} \overset{?}{=} \lim_{x \to 0} \frac{2x\sin(1/x) - \cos(1/x)}{1}$$

    has no limit for the right side, because $\cos(1/x)$ oscillates wildly near 0. We *cannot* conclude that $\lim_{x \to 0} \frac{x^2 \sin(1/x)}{x}$ does not exist. In fact,

$$\lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x \to 0} x \sin(1/x) = 0,$$

    by a basic application of the squeezing principle.

– There are also technical misreadings of the assumptions.
  * For Assumption 3 in L'Hôpital's Rule, we only need $\lim_{x\to a} \frac{f'}{g'}(x)$ to exist (in the extended real numbers), and this can occur even if $\lim_{x\to a} f'(x)$ or $\lim_{x\to a} g'(x)$ does not individually exist.
  * For Assumption 2 of L'Hôpital's Rule, you might wonder whether or not the condition "$g'(x) \neq 0$ on a deleted *neighbourhood* of a" is needed. The answer is "Yes, it is needed". (Look through the proof to find out why.)

  Almost all functions encountered in pre-rigorous calculus will not have such subtleties, even if they appear complicated. The easiest counter-examples that illustrate these subtleties have f being defined as an integral. Since we do not have a rigorous theory of integrals yet, we will instead provide an "$\frac{\infty}{\infty}$" L'Hôpital's Rule counter-example soon (see Example 10.1).

- Apart from technicalities, there is a more important *logical* and *conceptual* danger in the use of L'Hôpital's Rule.

  Consider the following (in)famous example where L'Hôpital's Rule is unnecessarily applied,

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{\text{L'Hôpital}}{=} \lim_{x\to 0} \frac{\sin'(x)}{1} = \lim_{x\to 0} \sin'(x). \tag{10.5}$$

  To proceed, we would need to know that $\sin'$ exists on some (deleted) neighbourhood around 0, and also check that we can take the $x \to 0$ limit of $\sin'$. While it may be "well-known" that $\sin' = \cos$ holds everywhere, and is continuous, we saw that this result needs significant work to prove rigorously. After substituting these results, we conclude that the left limit in (10.5) exists and is equal to $\sin'(0) = \cos(0) = 1$.

  Actually, the (two-sided) Baby L'Hôpital's Rule (Proposition 9.5) can be applied to this example,

$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{\sin'(0)}{(x)'(0)} = \sin'(0) = 1.$$

  Here, we only require knowledge of a single derivative, $\sin'(0)$.

  What did we actually learn from L'Hôpital rule? A closer look reveals that

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x - \sin 0}{x - 0} =: \sin'(0).$$

So the limit in question is equivalently the derivative of sin *at* 0*, by definition.* Therefore, the "usage" of the (Baby) L'Hôpital's Rule in (10.5) amounts to "using knowledge of $\sin'(0) = 1$ to prove $\sin'(0) = 1$".

Interestingly, we do not even need the derivative concept to prove that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \tag{10.6}$$

see mid-term exam.

Remark. In some textbooks, $\sin' = \cos$ is proved with the help of the limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$. (Sometimes, $\lim_{x\to 0} \frac{\sin x}{x} = 1$ is not rigorously proved!) In such an approach, using L'Hôpital's rule *with* $\sin' = \cos$ to "prove" $\lim_{x\to 0} \frac{\sin x}{x} = 1$ is a *circular argument.*

A related example is

$$\lim_{x\to 0} \frac{1-\cos x}{x} \overset{\text{L'Hôpital}}{=} \frac{\sin 0}{1} = 0. \tag{10.7}$$

This limit is actually a derivative,

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{(1-\cos x)-(1-\cos 0)}{x-0} \equiv \frac{d(1-\cos x)}{dx}\Big|_0 = -\cos'(0).$$

Independently, we proved that $\cos' = -\sin$, so

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \sin(0) = 0.$$

Actually, this limit can be computed without using the concept of differentiation, see mid-term exam.

Yet another related example is

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} \overset{\text{L'Hôpital}}{=} \lim_{x\to 0} \frac{1/(1+x)}{1} = 1. \tag{10.8}$$

This limit is precisely the derivative of $x \mapsto \ln(1+x)$ at $x = 0$,

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\ln(1+x)-\ln(1+0)}{x-0} =: \frac{d(\ln(1+x))}{dx}\Big|_0.$$

As before, we independently proved that

$$\frac{d(\ln(1+x))}{dx}\Big|_0 = \frac{1}{1+0} = 1.$$

If we put this result into the "use" of L'Hôpital's Rule in (10.8), we would have used knowledge of the derivative of $\ln(1+x)$ to compute its derivative at 0. (Tautological!) Once again, the limit (10.8) can be computed without differentiation — we already did this in (7.6) of Exercise 7.6.

Exercise 10.1. Let $n \geq 1$ and $x_0 \in \mathbb{R}$. Is it logical to compute the following limit using (iterated) L'Hôpital's Rule,

$$\lim_{h\to 0} \frac{(x_0+h)^n - x_0^n}{h} = \cdots = nx_0^{n-1}?$$

The above examples fit into a special case of the (Baby) L'Hôpital's Rule with g being $g(x) = x$. However, this kind of special case is *vacuous*. Here is the general reason:

– Note that the function $g : x \mapsto x - a$ satisfies those assumptions in the Baby L'Hôpital's Rule which involve g — these are $g(a) = 0, g'(a) \neq 0$ and $g(x) \neq 0$ on a deleted neighbourhood of a.

The remaining Baby L'Hôpital's Rule assumptions refer to f, and they are:

$$f(a) = 0, \qquad f \text{ differentiable at } a. \tag{10.9}$$

Now suppose we are given a function $\tilde{f}$, which is differentiable at a, i.e.,

$$\tilde{f}'(a) \overset{\text{definition}}{=} \lim_{x\to a} \frac{\tilde{f}(x) - \tilde{f}(a)}{x-a} \text{ exists in } \mathbb{R},$$

This is equivalent to the shifted function,

$$f(x) := \tilde{f}(x) - \tilde{f}(a),$$

satisfying the assumptions (10.9). Therefore, we may apply the Baby L'Hôpital's rule to f, g, obtaining the conclusion

$$\mathbb{R} \ni \ \tilde{f}'(a) \equiv \lim_{x\to a} \frac{\tilde{f}(x) - \tilde{f}(a)}{x-a} \equiv \lim_{x\to a} \frac{f(x)}{g(x)} = \underbrace{\frac{f'(a)}{g'(a)}}_{=1} = f'(a) = \tilde{f}'(a) \ \in \mathbb{R}. \tag{10.10}$$

This "conclusion" is a tautology!

– Perhaps the "real" L'Hôpital's Rule is better? Suppose $\tilde{f}$ is *continuously differentiable* on a neighbourhood of a. Then, with $f(x) = \tilde{f}(x) - \tilde{f}(a)$ and $g(x) = x - a$, it is straightforward to check (Exercise) that the assumptions needed in L'Hôpital's Rule are satisfied by f, g. Then, application of L'Hôpital's Rule to f, g leads to the tautologous "conclusion",

$$\mathbb{R} \ni \ \tilde{f}'(a) \equiv \lim_{x\to a} \frac{\tilde{f}(x) - \tilde{f}(a)}{x - a} \equiv \lim_{x\to a} \frac{f(x)}{g(x)}$$

$$= \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{\overbrace{\lim_{x\to a} f'(x)}^{f'(a)}}{\underbrace{\lim_{x\to a} g'(x)}_{1}} = f'(a) = \tilde{f}'(a) \ \in \mathbb{R}.$$

- In the next example, the denominator is not differentiable at the limit point $x = 0$, so we cannot use the Baby L'Hôpital's Rule. The real L'Hôpital's Rule gives

$$\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0^+} \frac{\cos x}{1/2\sqrt{x}} = \lim_{x\to 0^+} 2\sqrt{x}\cos x = 0. \tag{10.11}$$

  Once again, this limit can be evaluated quite easily *without* the L'Hôpital's Rule: simply use (10.6) (which does not use differentiation or L'Hôpital's Rule) and the product rule for limits,

$$\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0^+} \frac{\sin x}{x}\sqrt{x} = \lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \sqrt{x} = 1 \cdot 0 = 0.$$

- With some care, L'Hôpital's Rule can be iterated.

  If f, g fail Assumption 1 or 2, we are stuck.

  If f, g satisfy Assumptions 1,2,3, we can apply L'Hôpital's Rule.

  If f, g satisfy Assumptions 1,2, but we are unsure about Assumption 3,

  – If f', g' fail Assumptions 1 or 2, we are stuck.
  – If f', g' satisfy Assumptions 1,2,3, then L'Hôpital's Rule can be applied to f', g',

$$\lim_{x\to a^+} \frac{f'}{g'}(x) = \lim_{x\to a^+} \frac{(f')'}{(g')'}(x) \quad \text{exists.}$$

This also tells us that f, g satisfy Assumption 3: $\lim_{x\to a^+} \frac{f'}{g'}(x)$ exists in the extended reals. *Now*, we can also apply L'Hôpital's Rule for f, g, and conclude that

$$\lim_{x\to a^+} \frac{f}{g}(x) = \lim_{x\to a^+} \frac{f'}{g'}(x) \left(= \lim_{x\to a^+} \frac{(f')'}{(g')'}(x)\right) \quad \text{exists.} \tag{10.12}$$

– If f', g' only satisfy Assumptions 1,2, but we are unsure about Assumption 3,
  * If $(f')', (g')'$ fail Assumption 1 or 2, we are stuck.
  * If $(f')', (g')'$ satisfy Assumptions 1,2,3, we can apply L'Hôpital's Rule to $(f')', (g')'$ to deduce that

$$\lim_{x\to a^+} \frac{(f')'}{(g')'}(x) = \lim_{x\to a^+} \frac{((f')')'}{((g')')'}(x) \quad \text{exists.}$$

    Now we know that f', g' satisfy Assumption 3 in addition to Assumptions 1,2. So we can apply L'Hôpital's Rule to f', g' as well, to get

$$\lim_{x\to a^+} \frac{f'}{g'}(x) = \lim_{x\to a^+} \frac{(f')'}{(g')'}(x) \left(= \lim_{x\to a^+} \frac{(f')'}{(g')'}(x)\right) \quad \text{exists.}$$

    Now we know that f, g satisfy Assumption 3 in addition to Assumptions 1,2. So we can apply L'Hôpital's Rule to f, g as well, to get

$$\lim_{x\to a^+} \frac{f}{g}(x) = \lim_{x\to a^+} \frac{f'}{g'}(x) \left(= \lim_{x\to a^+} \frac{(f')'}{(g')'}(x) = \lim_{x\to a^+} \frac{((f')')'}{((g')')'}(x)\right) \quad \text{exists.}$$

  * If $(f')', (g')'$ satisfy Assumptions 1,2, but we are unsure about Assumption 3,
    · If $((f')')', ((g')')'$ fail Assumption 1 or 2, we are stuck.
    · If $((f')')', ((g')')'$ satisfy Assumptions 1,2,3, we can apply L'Hôpital's Rule...
    · ...
  * If $(f')', (g')'$ fail Assumption 1 or 2, we are stuck.

- Here is an amusing example where an attempt at iterated L'Hôpital's Rule never terminates:

$$\lim_{x\to+\infty}\frac{x}{\sqrt{1+x^2}}\overset{?}{=}\lim_{x\to+\infty}\frac{\sqrt{1+x^2}}{x}\overset{?}{=}\lim_{x\to+\infty}\frac{x}{\sqrt{1+x^2}}\overset{?}{=}\dots \tag{10.13}$$

  Although L'Hôpital's Rule fails here, either the limit exists or it doesn't. Here is one way to resolve the problem:

$$\lim_{x\to+\infty}\frac{x}{\sqrt{1+x^2}}=\lim_{x\to+\infty}\frac{1}{\sqrt{x^{-2}+1}}=\frac{1}{\sqrt{0+1}}=1.$$

  Above, we used the sequential definition of limit at $+\infty$, and the fact that for all positive sequences $x_1, x_2, \dots$ tending to $+\infty$, the reciprocal positive sequence $x_1^{-1}, x_2^{-1}, \dots$ converges to 0. Then continuity implies that

$$\lim_{n\to\infty}\frac{1}{\sqrt{x_n^{-2}+1}}=\frac{1}{\sqrt{\lim\limits_{n\to\infty}x_n^{-2}+1}}=1.$$

  A simpler version of (10.13) involves the hyperbolic tangent

$$\tanh(x)=\frac{\sinh(x)}{\cosh(x)},\qquad x\in\mathbb{R}.$$

  Recall from (8.9) that $\sinh'=\cosh$ and $\cosh'=\sinh$. Attempting L'Hôpital's Rule will result in a loop,

$$\lim_{x\to+\infty}\tanh x=\lim_{x\to+\infty}\frac{\sinh x}{\cosh x}\overset{?}{=}\lim_{x\to+\infty}\frac{\cosh x}{\sinh x}\overset{?}{=}\lim_{x\to+\infty}\frac{\sinh x}{\cosh x}\overset{?}{=}\dots$$

  The correct way to find the limit is to use $\lim\limits_{x\to\infty}e^{-2x}=0$ (Exercise), then

$$\lim_{x\to+\infty}\tanh x=\lim_{x\to+\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\lim_{x\to+\infty}\frac{1-e^{-2x}}{1+e^{-2x}}=\frac{1+0}{1-0}=1.$$

- Here are some examples where *iterated* L'Hôpital's Rule can be successfully applied.

  – Attempt L'Hôpital's Rule,

$$\lim_{x\to0}\frac{\ln(1+x)-x}{x^2}\overset{?}{=}\lim_{x\to0}\frac{(1+x)^{-1}-1}{2x} \tag{10.14}$$

The second limit is of the $\frac{0}{0}$ indeterminate form, and we need to check that it exists. A second attempt at L'Hôpital's Rule will resolve this issue. However, we could also simply observe that

$$\lim_{x\to 0}\frac{(1+x)^{-1}-1}{2x}=\lim_{x\to 0}\frac{\frac{1-(1+x)}{1+x}}{2x}=\lim_{x\to 0}-\frac{1}{2(1+x)}=-\frac{1}{2}.$$

So L'Hôpital's Rule does apply in (10.14), showing that

$$\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}=-\frac{1}{2}.$$

– Attempt L'Hôpital's Rule,

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}\overset{?}{=}\lim_{x\to 0}\frac{e^x-1}{2x}.$$

There is no need to use L'Hôpital's Rule again to assess the second indeterminate form. Just use the derivative-free Exercise 7.6 result), to deduce that the second limit exists and equals $\frac{1}{2}$. So the first limit exists and equals $\frac{1}{2}$ by L'Hôpital's Rule.

– Attempt L'Hôpital's Rule,

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}\overset{?}{=}\lim_{x\to 0}\frac{\sin x}{2x}. \tag{10.15}$$

Use (10.6) to conclude that the second limit exists and equals $\frac{1}{2}$. Then the first limit exists and equals $\frac{1}{2}$.

In the three example limits above, the second use of L'Hôpital's Rule is unnecessary. The first use can also be replaced by a direct use of Cauchy's MVT (unsurprisingly). In fact, these example limits can be computed without using derivatives, see Mid-Term Exam.

- Another example: L'Hôpital's Rule gives

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}\overset{?}{=}\lim_{x\to 0}\frac{1-\cos x}{3x^2}\overset{(10.15)}{=}\frac{1}{6}. \tag{10.16}$$

From the limit (10.16), we can work out Example 7.2.5 of Lou–Yang–Mei

(which is only discussed informally there),

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right) &= \lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}\\
&= \lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4}\cdot\left(\lim_{x\to 0}\frac{x}{\sin x}\right)^2\\
&= \lim_{x\to 0}\frac{x-\sin x}{x^3}\cdot\lim_{x\to 0}\frac{x+\sin x}{x}.1^2 && \text{(Eq. (10.6))}\\
&= \lim_{x\to 0}\frac{x-\sin x}{x^3}\left(1+\lim_{x\to 0}\frac{\sin x}{x}\right)\\
&= 2\lim_{x\to 0}\frac{x-\sin x}{x^3} && \text{(Eq. (10.6))}\\
&= \frac{1}{3}. && \text{(Example (10.16))}
\end{aligned}$$

Note that if we blindly apply iterated L'Hôpital's Rule after the first equality, we would get into a big mess.

Exercise 10.2. Compute the limit (10.16) using a triple angle trigonometric identity. (Such a identity could be proven with the MVT corollary 8.6.)

- Keep in mind that L'Hôpital's Rule relies on (Cauchy's) MVT, so it is no stronger than MVT. In a future lecture, we will see a richer consequence of MVT (occasionally via L'Hôpital's Rule) in the theory of Taylor series.

  For example, here is an apparent "Corollary" of L'Hôpital's Rule, which should probably be viewed as a Corollary of MVT.

  Corollary 10.1. *Let f be differentiable on a deleted neighbourhood of a. If $\lim_{x\to a} f'(x)$ exists and f is continuous at a, then f is differentiable at a, with*

$$f'(a)=\lim_{x\to a} f'(x).$$

  Proof. – [Proof with L'Hôpital's Rule.] Write $\tilde{f}(x)=f(x)-f(a)$ and $g(x)=x-a$, then, by definition of derivative,

$$f'(a)\equiv\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\tilde{f}(x)}{g(x)}.$$

As mentioned previously, g automatically satisfies the g part of the the L'Hôpital's Rule assumptions, so we check that $\tilde{f}$ satisfies them too.

* [Assumption 1.] Continuity of f at a implies

$$\lim_{x\to a} \tilde{f}(x) = \lim_{x\to a} f(x) - f(a) = f(a) - f(a) = 0.$$

* [Assumption 2.] f is differentiable on a deleted neighbourhood of a, so the shifted function $\tilde{f}$ is also differentiable there.
* [Assumption 3.] The limit

$$\lim_{x\to a} \frac{\tilde{f}'(x)}{g'(x)} = \lim_{x\to a} \frac{\tilde{f}(x)}{1} = \lim_{x\to a} f'(x)$$

  exists (in real numbers) by our assumption on f.

Therefore, L'Hôpital's Rule applies to $\tilde{f}, g$, and its conclusion is

$$f'(a) = \lim_{x\to a} \frac{\tilde{f}(x)}{g(x)} = \lim_{x\to a} \frac{\tilde{f}'(x)}{g'(x)} = \lim_{x\to a} \tilde{f}'(x) = \lim_{x\to a} f'(x) \quad (\text{in } \mathbb{R}).$$

– [Direct proof with MVT.] Let $\epsilon > 0$ be given. Since $L = \lim_{x\to a} f'(x)$ exists, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \;\Rightarrow\; |f'(x) - L| < \epsilon. \tag{10.17}$$

Here, we choose δ small enough, so that $(a-\delta, a) \cup (a, a+\delta)$ lies within the deleted neighbourhood on which f is differentiable.
Let $x \in (a-\delta, a) \cup (a+\delta)$.

* [$x \in (a, a+\delta)$.] By assumption, f is differentiable (thus continuous) on the subinterval $(a, x]$. Also by assumption, f is continuous at a. So f is continuous on $[a, x]$ and differentiable in its interior. By Lagrange's MVT, there exists $\gamma \equiv \gamma(x) \in (a, x)$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(\gamma).$$

* [$x \in (a-\delta, a)$.] Similarly, there exists $\gamma \equiv \gamma(x) \in (x, a)$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(\gamma).$$

In either case, $|\gamma(x) - a| < |x - a| < \delta$ holds, so (10.17) applies, giving

$$0 < |x - a| < \delta \;\Rightarrow\; \left| \frac{f(x) - f(a)}{x - a} - L \right| = |f'(\gamma(x)) - L| < \epsilon.$$

This shows that

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = L = \lim_{x \to a} f'(x)$$

exists in $\mathbb{R}$. But the left side is precisely $f'(a)$ by definition.

□

Exercise 10.3. Use the fact that $\lim_{x \to 0} \frac{\ln(1+x)-x}{x^2} = -\frac{1}{2}$ to show that the function

$$f : x \mapsto \begin{cases} (1+x)^{1/x}, & x \in \mathbb{R} \setminus \{0\}, \\ e, & x = 0, \end{cases}$$

is differentiable at 0. (Note that we showed continuity of f at 0 in (6.3).)

Remark. Exercise 10.3 gives a L'Hôpital's Rule-free (or even derivative-free) way to establish and compute the limit

$$\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x} =: f'(0).$$

This limit was analyzed in Example 7.2.6 of Lou–Mei–Yang, using iterated L'Hôpital's Rule in two different ways, giving two different answers. You are asked to locate the error there. This illustrates some dangers when blindly and unnecessarily using L'Hôpital's Rule.

Exercise 10.4. Use Corollary 10.1 and $\lim_{x \to 0} \frac{\ln(1+x)-x}{x^2} = -\frac{1}{2}$ to show that the function f in Exercise 10.3 is continuously differentiable at $x = 0$.

- If you have already informally learned about Taylor series for elementary functions, you may have noticed that the above examples where L'Hôpital's Rule could be applied (non-tautologically), are "reverse engineered" from Taylor series ideas. Specifically, they have (shifted) monomials like $x, x - a, x^2, x^3$ as denominators. In such examples, the Taylor series provides much more information than just a single limit at a single point. We will discuss this theory systematically in a future lecture. For

now, let us just give an indication of how L'Hôpital's Rule and Taylor series are related.

Consider the situation where f is *twice*-differentiable at x_0, i.e.,

- f is differentiable on some neighbourhood W of x_0, and
- f' is differentiable at x_0.

First, continuity of f at x_0 means: Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \;\Rightarrow\; \big|f(x) - T_0(x)\big| < \epsilon. \tag{10.18}$$

Informally, T_0 is the "best constant approximation" to f at x_0.

Furthermore, differentiability of f at x_0 means that

$$f'(x_0) \equiv \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{f(x) - T_0(x)}{x - x_0} \quad \text{exists in } \mathbb{R},$$

i.e.,

$$\lim_{x\to x_0} \left| \frac{(f - T_0)(x)}{x - x_0} - f'(x_0) \right| = 0.$$

Equivalently, in ϵ-δ language: Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \;\Rightarrow\; \big|f(x) - T_1(x)\big| < \epsilon |x - x_0|, \tag{10.19}$$

where

$$T_1(x) := T_0(x) + f'(x_0)(x - x_0) = f(x_0) + f'(x_0)(x - x_0).$$

We note that

$$T_1(x_0) = f(x_0), \qquad T_1'(x) = f'(x_0). \tag{10.20}$$

Compared to T_0, the function T_1 has the same value and the same derivative as f at x_0. Informally, T_1 the "best linear approximation" to f at x_0.

Furthermore, second-differentiability of f at x_0, i.e., differentiability of f' at x_0, means

$$(f')'(x_0) \equiv \lim_{x\to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} \quad \text{exists in } \mathbb{R}. \tag{10.21}$$

Because of (10.20) and (10.19) (deduced from first-differentiability), the numerator of (10.21) is

$$f'(x) - f'(x_0) = (f - T_1)'(x), \qquad \lim_{x \to x_0} (f - T_1)(x) = 0.$$

The denominator of (10.21) is

$$x - x_0 = \frac{d\big(\frac{1}{2}(x - x_0)^2\big)\big)}{dx}\Big|_{x_0}, \qquad \lim_{x \to x_0} \frac{1}{2}(x - x_0)^2 = 0.$$

The assumptions of L'Hôpital's Rule are satisfied by the functions $f - T_1$ and $x \mapsto (x - x_0)^2$, so we may conclude that

$$\lim_{x \to x_0} \frac{f(x) - T_1(x)}{\frac{1}{2}(x - x_0)^2} = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} \overset{\text{definition}}{=} (f')'(x_0),$$

i.e.,

$$\lim_{x \to x_0} \left| \frac{(f - T_1)(x)}{\frac{1}{2}(x - x_0)^2} - (f')'(x_0) \right| = 0.$$

Equivalently, in ϵ-δ language: Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \;\Rightarrow\; \big|f(x) - T_2(x)\big| < \epsilon |x - x_0|^2, \tag{10.22}$$

where

$$\begin{aligned} T_2(x) &:= T_1(x) + \frac{1}{2}(f')'(x_0)(x - x_0)^2 \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}(f')'(x_0)(x - x_0)^2. \end{aligned} \tag{10.23}$$

We note that

$$T_2(x_0) = f(x_0), \qquad T_2'(x_0) = f'(x_0), \qquad (T_2')'(x) = (f')'(x_0). \tag{10.24}$$

Compared to T_1, the function T_2 has the same value, first derivative, and second derivative as f at x_0. Informally, T_2 is the "best quadratic approximation" to f at x_0.

We expect that if, furthermore, f is n-times differentiable, then we should be able to iteratively construct a "best degree-n polyomial approximation" to f at x_0. We will explore this idea in a later lecture.

- Here is an "$\frac{\infty}{\infty}$" candidate for L'Hôpital's Rule.

$$f(x) = 2x - \cos x - \frac{\sin x}{x}, \qquad g(x) = 2x - \cos x, \qquad x > 0,$$

have limits $+\infty$ as $x \to +\infty$ (why?). For $x > 0$, their derivatives are

$$\begin{aligned} f'(x) &= 2 + \sin x - \frac{\cos x}{x} + \frac{\sin x}{x^2}, \\ g'(x) &= 2 + \sin x. \end{aligned}$$

Because of the oscillating $\sin x$ term, the bounded functions f', g' cannot have a limit as $x \to +\infty$. Nevertheless,

$$\lim_{x\to+\infty} \frac{f'}{g'}(x) = 1 + \lim_{x\to+\infty} \frac{-\frac{\cos x}{x} + \frac{\sin x}{x^2}}{2 + \sin x} \tag{10.25}$$

The numerator has limit 0 as $x \to \infty$ (Squeezing principle), while the denominator stays bounded between -1 and 3. So the limit on the right side of (10.25) is 0. Thus the *ratio* of derivatives on the left side of (10.25) has a well-defined limit,

$$\lim_{x\to+\infty} \frac{f'}{g'}(x) = 1.$$

This result coincides with

$$\lim_{x\to+\infty} \frac{f}{g}(x) = \lim_{x\to\infty} \frac{2 - \frac{\cos x}{x} - \frac{\sin x}{x^2}}{2 - \frac{\cos x}{x}} = \frac{2}{2} = 1.$$

This gives us some confidence that there exists an "$\frac{\infty}{\infty}$" L'Hôpital's Rule.

- However, here is an "$\frac{\infty}{\infty}$ counter-example" to an imprecisely stated L'Hôpital's Rule.

  Example 10.1 (Stolz).

$$f(x) = x + \sin x \cos x, \qquad g(x) = f(x)e^{\sin x}, \qquad x \in \mathbb{R}.$$

  Certainly,

$$\lim_{x\to0} f(x) = +\infty = \lim_{x\to0} g(x),$$

  and we can ask whether

$$\lim_{x\to\infty} \frac{f}{g}(x)$$

makes sense.

A blind belief in L'Hôpital's Rule would lead us to consider the derivatives

$$f'(x) = 1 + \cos^2 x - \sin^2 x = 2\cos^2 x,$$
$$g'(x) = e^{\sin x}\big(f'(x) + f(x)\cos x\big) = e^{\sin x}(\cos x)(2\cos x + x + \sin x \cos x),$$

which exist everywhere. Note that due to the oscillating $\cos x$, the limits of f' and g' at $+\infty$ do not individually exist. The *ratio* of derivatives,

$$\frac{f'}{g'}(x) = \frac{2\cos^2 x}{e^{\sin x}(\cos x)(2\cos x + x + \sin x \cos x)},$$

is defined everywhere except when

$$g'(x) = 0 \;\Leftrightarrow\; \cos x = 0 \;\text{ or }\; 2\cos x + x + \sin x \cos x = 0.$$

As $|\sin x|, |\cos x| \le 1$, we may conclude that $2\cos x + x + \sin x \cos x > 0$ holds on $(3, +\infty)$. Therefore, $\frac{f'}{g'}$ is defined on the set

$$E = (3, +\infty) \setminus \Big\{n\pi + \frac{\pi}{2} \;:\; n \ge 2\Big\}.$$

Note that we cannot consider $\frac{f'}{g'}$ to be defined on any full interval $(a, +\infty)$, because g' always vanishes on some set of isolated points in such an interval.

Nevertheless, since E is unbounded above, we can still consider the limit of $\frac{f'}{g'}$ at $+\infty$,

$$\lim_{x\to+\infty\,:\,x\in E} \frac{f'}{g'}(x) = \lim_{x\to+\infty\,:\,x\in E} \frac{2\cos x}{e^{\sin x}(2\cos x + x + \sin x\cos x)}.$$

We have $\left|\frac{2\cos x}{e^{\sin x}}\right| \le 2e$, while

$$|2\cos x + x + \sin x\cos x| \ge |x| - |2\cos x + \sin x\cos x| > x - 3. \qquad (x \in E,\ \text{so } x > 3)$$

So

$$\left|\frac{f'}{g'}(x)\right| \le \frac{2e}{x-3}, \qquad x \in E.$$

As E is unbounded above, given $\epsilon > 0$, we can always find some $x \in E$ such that $\left|\frac{f'}{g'}(x)\right| < \epsilon$. Therefore, the *ratio of derivatives* has a limit at $+\infty$,

$$\lim_{x \to +\infty \,:\, x \in E} \frac{f'}{g'}(x) = 0.$$

Actually, we can improve the situation. The set E is the interval $(3, +\infty)$ with those points where $\cos x = 0$ deleted. Cancelling $\cos x$ from numerator and denominator, we could define the function

$$h(x) = \frac{2 \cos x}{e^{\sin x}(2 \cos x + x + \sin x \cos x)}, \qquad x \in (3, +\infty),$$

is defined on a full interval and coincides with $\frac{f'}{g'}$ on $E \subseteq (3, +\infty)$. The same calculation as above gives

$$\lim_{x \to +\infty} h(x) = 0.$$

An "$\frac{\infty}{\infty}$" L'Hôpital's Rule suggests that

$$\lim_{x \to +\infty} \frac{f}{g}(x) \stackrel{?}{=} \lim_{x \to +\infty \,:\, x \in E} \frac{f'}{g'}(x) = \lim_{x \to +\infty} h(x) = 0. \tag{10.26}$$

However, the $x \to +\infty$ limit of $\frac{f}{g}$ does not exist! For $x > 1$, we have $g(x) > 0$, so $\frac{f}{g}(x)$ is well-defined,

$$\frac{f}{g}(x) = e^{-\sin x}, \qquad x > 1.$$

It is straightforward to show that $e^{-\sin x}$ does not have a limit at $+\infty$ (Exercise). Even if we work within the smaller domain E, the $+\infty, x \in E$ limit of $e^{-\sin x}$ still does not exist (why?).

The lesson from Example 10.1 is: For a correct L'Hôpital's Rule, it is not enough to assume that $g'(x) \neq 0$ holds on some set E for which a is a limit point. Instead, we must really impose $g'(x) \neq 0$ on some deleted (one-sided or two-sided) *neighbourhood* of a, as in Assumption 2.