11 Lectures 21–22

- The issue with the Stolz "counterexample' is the fact that the restriction of g' to *any* interval $(a, +\infty)$ always has zeroes. This will cause problems when trying to apply the quotient form (9.19) of Cauchy's MVT in the proof of L'Hôpital's Rule.

  Recall that this form of Cauchy's MVT holds for functions x, y which are differentiable on some open interval I, with $x'(t) \neq 0$ on I. Then for any $a, b \in I$ with $a < b$,

  $$\exists c \in (a, b) \text{ such that } \frac{y(b) - y(a)}{x(b) - x(a)} = \frac{y'(c)}{x'(c)}. \tag{11.1}$$

  Here is an example illustrating why it is necessary to assume $x'(t) \neq 0$ on the whole interval:

  $$x(t) = t^3, \qquad y(t) = t^2, \qquad t \in [-1, 1]. \tag{11.2}$$

  This has

  $$x'(t) = 3t^2, \qquad y'(t) = 2t, \qquad t \in (-1, 1),$$

  so $x'(0) = 0$. We can write y in terms of x,

  $$\tilde{y}(x) = |x|^{2/3}.$$

  The graph of $\tilde{y} \equiv \tilde{y}(x)$ has a "cusp" at $(0, 0)$, where $\tilde{y}'$ becomes ill-defined. Now suppose $c \in (-1, 1)$ satisfies

  $$\frac{y'(c)}{x'(c)} = \frac{y(1) - y(-1)}{x(1) - x(-1)} = \frac{0}{2} = 0.$$

  This would require $y'(c) = 2c = 0$, so $c = 0$. But $x'(c) = x'(0) = 0$, so $\frac{y'(c)}{x'(c)}$ is ill-defined. So Cauchy's MVT conclusion (11.1) fails.

  Remark. The product form of Cauchy's MVT, Eq. (9.18), does hold for the example (11.2):

  $$0 = \big(x(1) - x(-1)\big)y'(0) = \big(y(1) - y(-1)\big)x'(0) = 0.$$

- Let us state and prove a correct "$\frac{*}{\infty}$" variant of L'Hôpital's Rule.

  Theorem 11.1 (L'Hôpital's Rule, $\frac{*}{\infty}$ version)**.** *Let $a \in \{-\infty\} \sqcup \mathbb{R}$ and f, g be real-valued functions satisfying*

1′. $\lim_{x\to a^+} g(x) \in \{-\infty, +\infty\}$;

as well as Assumptions 2,3 of Theorem 9.6. Then

$$\lim_{x\to a^+} \frac{f}{g}(x) = \lim_{x\to a^+} \frac{f'}{g'}(x). \tag{11.3}$$

Similar statements hold for left-sided and two-sided limits.

Proof. We explain the case of right-sided limits $\lim_{x\to a^+}$, with $a \in \mathbb{R}$, first. We may assume that Assumption 1′ on g is $\lim_{x\to a^+} g(x) = +\infty$. (Otherwise consider $-g$.) Note that this implies that g is strictly positive on some interval (a, b), which we may assume is the interval on which $g' \neq 0$ (given by Assumption 2).

– [Case $\lim_{x\to a^+} \frac{f'}{g'}(x) = L \in \mathbb{R}$ in Assumption 3.] Given $\epsilon > 0$, there exists some subinterval $(a, y(\epsilon)) \subseteq (a, b)$ with $y \equiv y(\epsilon) < b$, such that

$$L - \epsilon < \frac{f'}{g'}(c) < L + \epsilon, \qquad \forall c \in (a, y(\epsilon)) \subseteq (a, b). \tag{11.4}$$

For each $x \in (a, y(\epsilon))$, Assumption 2 allows us to apply Cauchy's MVT to $f|_{[x,y]}, g|_{[x,y]}$, to get $\gamma \equiv \gamma(x) \in (x, y)$ satisfying

$$\frac{f'(\gamma(x))}{g'(\gamma(x))} = \frac{f(y) - f(x)}{g(y) - g(x)}, \qquad x \in (a, y) \equiv (a, y(\epsilon)).$$

We are allowed to divide by $g(x) \neq 0$, then after a rearrangement,

$$\frac{f(x)}{g(x)} = \frac{f(y)}{g(x)} + \frac{f'(\gamma(x))}{g'(\gamma(x))}\left(1 - \frac{g(y)}{g(x)}\right), \qquad x \in (a, y(\epsilon)). \tag{11.5}$$

Note that by Assumption 1′, there exists some subinterval $(a, \tilde{\delta}(\epsilon)) \subseteq (a, y(\epsilon))$ such that $1 - \frac{g(y)}{g(x)} > 0$ holds for all $x \in (a, \tilde{\delta}(\epsilon))$. Also, inequality (11.4) holds with $c = \gamma(x)$, for all $x \in (a, \tilde{\delta}(\epsilon))$. Substitute this inequality into (11.5) to get

$$\begin{aligned} &x \in (a, \tilde{\delta}(\epsilon)) \\ &\Rightarrow\ (L-\epsilon)\underbrace{-\frac{(L-\epsilon)g(y)}{g(x)} + \frac{f(y)}{g(x)}}_{=:h^-(x)} < \frac{f(x)}{g(x)} < (L+\epsilon)\underbrace{-\frac{(L+\epsilon)g(y)}{g(x)} + \frac{f(y)}{g(x)}}_{=:h^+(x)}. \end{aligned} \tag{11.6}$$

By Assumption $1'$, we have

$$\lim_{x\to a^+} h^{\pm}(x) = 0,$$

which means that we can find a further subinterval $(a, \delta(\epsilon)) \subseteq (a, \tilde{\delta}(\epsilon))$ such that

$$x \in (a, \delta(\epsilon)) \;\Rightarrow\; |h^{\pm}(x)| < \epsilon.$$

Then (11.6) becomes

$$x \in (a, \delta(\epsilon)) \;\Rightarrow\; L - 2\epsilon < \frac{f(x)}{g(x)} < L + 2\epsilon.$$

As $\epsilon > 0$ was arbitrary, we have shown that $\lim_{x\to a^+} \frac{f}{g}(x)$ exists, and equals L.

– [Case $\lim_{x\to a^+} \frac{f'}{g'}(x) \in \{-\infty, +\infty\}$ in Assumption 3.] Similar. This is left as an Exercise.

The case of left-sided limits, $\lim_{x\to b^-}$, with $b \in \mathbb{R}$, is similar.

For the $\lim_{x\to+\infty}$ case, we use Exercise 9.5 to argue that $\tilde{f}(t) = f(t^{-1})$ and $\tilde{g}(t) = g(t^{-1}$ satisfy Assumptions $1', 2, 3$ for $a = 0$. Then apply the right-sided L'Hôpital's Rule for proved earlier, together with Exercise 9.5 again.

The $\lim_{x\to-\infty}$ case is similar.

□

Remark 2. Theorem 11.1 is often stated with the unnecessary extra assumption that $\lim_{x\to a^+} f(x) \in \{-\infty, +\infty\}$, and is then called the "$\frac{\infty}{\infty}$" L'Hôpital's Rule. Some authors also write Assumption $1'$ as

$$\lim_{x\to a^+} |g(x)| = \infty. \tag{11.7}$$

By itself, (11.7) is weaker that our Assumption $1'$. But under Assumption 2, g is strictly monotone on (a, b) (see Exercise 9.4), and we may deduce that (11.7) implies our Assumption $1'$. (Exercise.) In other words, the combined Assumptions $1', 2$ are equivalent to (11.7) combined with Assumption 2.

Of course, if f is a *bounded* function[22], then Assumption $1'$, or even just the weaker assumption (11.7)), is already enough to show that

$$\lim_{x\to a^+} \frac{f}{g}(x) = 0.$$

(Exercise.) So the $\frac{*}{\infty}$ L'Hôpital's Rule only says something interesting (via the derivatives f', g'), when *both* f, g become *unbounded* near a.

Remark 3. Suppose f, g satisfy the all the assumptions of the left-sided $\frac{*}{\infty}$ L'Hôpital's Rule. Let us write

$$L := \lim_{x\to b^-} \frac{f'}{g'}(x) \in \{-\infty\} \sqcup \mathbb{R} \sqcup \{+\infty\}.$$

The conclusion of L'Hôpital's Rule is

$$\lim_{x\to b^-} \frac{f}{g}(x) = L \tag{11.8}$$

- Suppose $L \neq 0$. Since $\lim_{x\to b^-} g(x) = \pm\infty$ by Assumption $1'$, Eq. (11.8) implies that $\lim_{x\to b^-} f(x) = \pm\infty$. So we are really in an "$\frac{\infty}{\infty}$" version of L'Hôpital's Rule.
- The contrapositive version of the above implication is: When $\frac{*}{\infty}$ L'Hôpital's Rule holds but $\lim_{x\to b^-} f(x) \neq \pm\infty$, we must have $L = 0$.

  Note that $\lim_{x\to b^-} f(x) \neq \pm\infty$ occurs in two cases: Either f is bounded (this is uninteresting by Remark 2), or f becomes unbounded near b and oscillates wildly between arbitrarily large positive and negative values.

  Example 11.1. Consider

  $$f(x) = \sqrt{x}\sin(\ln(x)), \qquad g(x) = x, \qquad x > 0.$$

  This f oscillates wildly as $x \to +\infty$. Assumptions $1', 2$ are satisfied, so by the above discussion, the only possible value of $L = \lim_{x\to+\infty} \frac{f'}{g'}(x)$ is 0. Indeed,

  $$L = \lim_{x\to+\infty} \frac{f'}{g'}(x) = \lim_{x\to+\infty} \frac{\sin(\ln x) + 2\cos(\ln x)}{2\sqrt{x}} = 0.$$

[22]Locally bounded, on some neighbourhood of a, is also OK.

So we are in a $\frac{*}{\infty}$ (but not $\frac{\infty}{\infty}$) L'Hôpital's Rule situation, and Theorem 11.1 gives the conclusion

$$\lim_{x\to+\infty} \frac{f}{g}(x) = 0.$$

This is consistent with a direct Squeezing principle calculation.

- The "$\frac{*}{\infty}$" L'Hôpital's Rule at a tells us about the relative behaviour of unbounded functions which blow up at a.

  Visualization: Consider the parametrized curve picture,

  $$\gamma : (a, b) \to \big(x(t), y(t)\big).$$

  We consider the left-sided L'Hôpital's Rule for y, x. So Assumption 1′ says that

  $$\lim_{t\to b^-} x(t) = \pm\infty,$$

  where $b \in \mathbb{R} \sqcup \{+\infty\}$. Assumption 2 says that the curve is differentiable with $x'(t) \neq 0, t \in (a, b)$, and the ratio $\frac{y'(t)}{x'(t)}$ is precisely the slope $\tilde{y}'(x(t)) = \frac{d\tilde{y}}{dx}\Big|_{x(t)}$ of the tangent to the graph of the function

  $$\tilde{y} : x(t) \mapsto y(t), \qquad t \in (a, b).$$

  (This was derived in (9.11).) Assumption 3 states that these slopes tend to some limit $L \in \{-\infty\} \sqcup \mathbb{R} \sqcup \{+\infty\}$.

  Consider the $L \in \mathbb{R}$ case. Given $0 < \epsilon < L/2$, let $C_{2\epsilon}$ be the (double) cone with vertex $(0, 0)$ and boundary slopes $L - 2\epsilon, L + 2\epsilon$. There is a subinterval $(c, b) \subseteq (a, b)$ such that the slope of the curve stays between $L - \epsilon$ and $L + \epsilon$ for all $t \in (c, b)$. An intuitive consequence is: for each $t \in (c, b)$, the curve is "squeezed" inside the cone $C_\epsilon(t)$ with vertex $(x(t), y(t))$ and boundary slopes $L - \epsilon, L + \epsilon$. This was justified with MVT in lectures.

  The vertex $(x(c), y(c))$ might lie outside $C_{2\epsilon}$. But by Assumption 1′, $x(t)$ tends to $\pm\infty$ as $t \to b^-$, and the boundary slopes of $C_\epsilon(c)$ are in between the boundary slopes of $C_{2\epsilon}$. $C_\epsilon(c)$ will eventually "enter" $C_{2\epsilon}$ from some time $d \in (c, b)$ onwards. This implies that from time d onwards, the tail of the curve stays inside $C_{2\epsilon}$. All points in this tail satisfy $L - 2\epsilon < \frac{y(t)}{x(t)} < L + 2\epsilon$. This shows that $\frac{y(t)}{x(t)} \to L$ as $t \to b^-$.

Note: For $L = 0$, the curve may be oscillating in the y-direction as $t \to b$. Nevertheless, the curve still eventually gets squeezed inside a cone with boundary slopes -2ϵ and 2ϵ.

The $L = \pm\infty$ case can be visualized similarly. Here, we must have $y(t) \to \pm\infty$ as $t \to b^-$, as mentioned in Remark 3. Also, $y'(t)$ remains nonzero for t close to b^-. So we could exchange the roles of x, y, so that we are in the $L = 0$ situation visualized above.

Example 11.2. Let $\alpha > 0$ and write $m = \lfloor \alpha \rfloor$. We attempt to iterate the "$\frac{\infty}{\infty}$" L'Hôpital's Rule $m + 1$ times,

$$\lim_{x\to+\infty} \frac{x^\alpha}{e^x} \stackrel{?}{=} \lim_{x\to+\infty} \frac{\alpha x^{\alpha-1}}{e^x} \stackrel{?}{=} \ldots \stackrel{?}{=} \lim_{x\to+\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-m)x^{\alpha-m-1}}{e^x}$$

Since $\alpha - m - 1 < 0$, the last limit exists and equals 0. So we conclude that

$$\lim_{x\to+\infty} \frac{x^\alpha}{e^x} = 0.$$

This implies the well-known statement that a monomial (thus a polynomial) grows more slowly than an exponential.

Example 11.3. Recall that $\ln : (0, \infty) \to \mathbb{R}$ has range $\mathbb{R}$ and is monotone increasing. It follows that $\ln x$ tends to $+\infty$ as x tends to $+\infty$. We may apply the "$\frac{\infty}{\infty}$" L'Hôpital's Rule to deduce that

$$\lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{1} = 0. \tag{11.9}$$

Actually, substituting $y = \ln x$, we have

$$\lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{y\to+\infty} \frac{y}{e^y},$$

so (11.9) is a special case of Example 11.2 of L'Hôpital's Rule. Alternatively, we could use the power series

$$e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!} > \frac{y^2}{2} \;\Rightarrow\; 0 < \frac{y}{e^y} < \frac{y}{\frac{y^2}{2}} = \frac{2}{y},$$

then apply the Squeezing principle. One says that a logarithm grows more slowly than a polynomial.

Applying the continuous exponential function to (11.9), we also deduce that

$$\lim_{x\to+\infty} x^{\frac{1}{x}} = \lim_{x\to+\infty} e^{\frac{\ln x}{x}} = e^{\lim\limits_{x\to+\infty}\frac{\ln x}{x}} = e^0 = 1. \tag{11.10}$$

Example 11.4 (Modified Example 7.2.8, 7.2.7 of Lou–Mei–Yang). Let f be differentiable on $(a, +\infty)$.

– Suppose

$$\lim_{x\to+\infty} (f' + f)(x) \text{ exists in } \mathbb{R}.$$

Observe that

$$(f' + f)(x) = \frac{\left.\frac{d(e^x f(x))}{dx}\right|_x}{\left.\frac{d(e^x)}{dx}\right|_x}$$

has a real limit at $+\infty$, so Assumption 3 of L'Hôpital's Rule holds for $e^x f(x), e^x$. It is easily checked that Assumptions $1'$, 2 also hold. So we conclude that

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{e^x f(x)}{e^x} = \lim_{x\to+\infty} (f' + f)(x) \text{ exists in } \mathbb{R}.$$

– Suppose

$$\lim_{x\to+\infty} xf'(x) \in \mathbb{R} \setminus \{0\}.$$

Setting $g(x) = \ln x$, we attempt a "$\frac{*}{\infty}$" L'Hôpital's Rule,

$$\lim_{x\to+\infty} \frac{f(x)}{\ln x} = \lim_{x\to+\infty} xf'(x) \text{ exists in } \mathbb{R} \setminus \{0\}.$$

Multiplying by $\ln(x)$, it follows that

$$\lim_{x\to+\infty} f(x) \in \{-\infty, +\infty\}.$$

Exercise 11.1 (Modified Example 7.2.7 of Lou–Mei–Yang). Let f be differentiable on $(a, +\infty)$.

– Suppose $\lim\limits_{x\to+\infty} f(x)$ exists in $\mathbb{R}$. Explain whether we can apply L'Hôpital's Rule to deduce

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{xf(x)}{x} \overset{?}{=} \lim_{x\to+\infty} \frac{f(x) + xf'(x)}{1} = \lim_{x\to+\infty} f(x) + \lim_{x\to+\infty} xf'(x)$$
$$\Rightarrow \lim_{x\to+\infty} xf'(x) = 0.$$

– Suppose f is a bounded function, and f' is monotone. Show that $\lim_{x\to+\infty} f(x)$ exists in $\mathbb{R}$, and that

$$\lim_{x\to+\infty} xf'(x) = 0. \tag{11.11}$$

Example 11.5 (Example 7.2.11 of Lou–Mei–Yang). Suppose f is a twice-differentiable solution to the differential equation

$$(f')'(x) + xf'(x) + f(x) = 0, \qquad x \in \mathbb{R}. \tag{11.12}$$

Below, we will prove that f must satisfy the "boundary condition",

$$\lim_{x\to+\infty} f(x) = 0 = \lim_{x\to+\infty} f'(x).$$

Rewrite (11.12) as

$$\big(f'(x) + xf(x)\big)' = 0, \qquad x \in \mathbb{R}.$$

So there is a constant $c \in \mathbb{R}$ such that $f'(x) + xf(x) = c, \forall x \in \mathbb{R}$. (Exercise.) Observe that

$$\frac{f'(x) + xf(x)}{x} = \frac{(f'(x) + xf(x))e^{x^2/2}}{xe^{x^2/2}} = \frac{\left.\frac{d(e^{x^2/2}f(x))}{dx}\right|_x}{\left.\frac{d(e^{x^2/2})}{dx}\right|_x}.$$

This has limit $\lim_{x\to+\infty} \frac{c}{x} = 0$. So we may apply L'Hôpital's Rule to deduce that

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{f(x)e^{x^2/2}}{e^{x^2/2}} \overset{\text{L'Hôpital}}{=} 0.$$

Similarly, because of the identity

$$\frac{-f(x)}{x} \overset{(11.12)}{=} \frac{((f')'(x) + xf'(x))e^{x^2/2}}{xe^{x^2/2}} = \frac{\left.\frac{d(e^{x^2/2}f'(x))}{dx}\right|_x}{\left.\frac{d(e^{x^2/2})}{dx}\right|_x},$$

which has limit 0 at $+\infty$, we may apply L'Hôpital's Rule to deduce that

$$\lim_{x\to+\infty} f'(x) = \lim_{x\to+\infty} \frac{e^{x^2/2}f'(x)}{e^{x^2/2}} \overset{\text{L'Hôpital}}{=} 0.$$

- There is a discrete version of the $\frac{*}{\infty}$ L'Hôpital's Rule at $+\infty$, involving sequences, with a similar proof.

  Theorem 11.2 (Stolz–Cesàro, $\frac{*}{\infty}$ version)**.** *Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be real sequences. If*

  1′. $\lim_{n\to\infty} y_n \in \{-\infty, +\infty\}$*;*

  2. $y_1, y_2, \ldots$ *is strictly monotone;*

  then

  $$\underline{\lim_{n\to\infty}} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} \leq \underline{\lim_{n\to\infty}} \frac{x_n}{y_n} \leq \overline{\lim_{n\to\infty}} \frac{x_n}{y_n} \leq \overline{\lim_{n\to\infty}} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}. \tag{11.13}$$

  Suppose the following also holds,

  3. $\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$ *exists in the extended real numbers.*

  Then $\lim_{n\to\infty} \frac{x_n}{y_n}$ *exists in the extended real numbers, and equals* $\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$.

  Comments:

  - The three assumptions in Theorem 11.2 are precisely the sequential analogues of the three assumptions in Theorem 11.1:
    1′. Obvious.
    2. Relationship to monotonicity was explained in Remark 2.
    3. Think of $x_{n+1} - x_n = \frac{x_{n+1}-x_n}{(n+1)-n}$ as a "discrete derivative" of the x sequence at n. So $\frac{x_{n+1}-x_n}{y_{n+1}-y_n}$ is the ratio of "discrete derivatives" of the two sequences. (The denominator is non-zero by Assumption 2.)
  - As mentioned at the end of Lectures 7–8, we can consider the liminf and limsup of possibly-unbounded sequences. These always exist in the extended real numbers; e.g., if there is no upper bound, then the relevant (extended real-valued) sequence of supremums stays at $+\infty$, so it has limit $+\infty$. You may wish to write out the explicit meaning of $\underline{\lim_{n\to\infty}}$, $\overline{\lim_{n\to\infty}}$ in the extended real numbers setting.

Furthermore,

$$\underline{\lim}_{n\to\infty} x_n \le \overline{\lim}_{n\to\infty} x_n,$$

$$\text{If } x_n \le y_n \ \forall n \ge 1, \text{ then } \quad \underline{\lim}_{n\to\infty} x_n \le \underline{\lim}_{n\to\infty} y_n, \quad \text{and} \quad \overline{\lim}_{n\to\infty} x_n \le \overline{\lim}_{n\to\infty} y_n. \tag{11.14}$$

– It is possible that some of the y_n are zero, so some of the $\frac{x_n}{y_n}$ are ill-defined (even in the extended real number sense). But we are assuming that $\lim_{n\to\infty} y_n = \pm\infty$, so some tail of y_n will avoid 0, and the liminf, limsup, lim only depend on the tail.

Proof. By writing $\frac{-x_n}{-y_n}$ instead of $\frac{x_n}{y_n}$, we may assume that $\lim_{n\to\infty} y_n = +\infty$ case (thus y_n is strictly monotone increasing in Assumption 2).

Write

$$\ell = \underline{\lim}_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \lim_{n\to\infty} \inf_{k\ge n} \frac{x_{k+1} - x_k}{y_{k+1} - y_k}.$$

If $\ell = -\infty$, the liminf inequality in (11.13) is automatic, so suppose $\ell \in \mathbb{R} \sqcup \{+\infty\}$. By $\lim_{n\to\infty} y_n = +\infty$, there exists $N_1 \ge 1$ such that $y_k > 0$ for all $k \ge N_1$. For any $c_- < \ell$, there exists $N_2 \ge N_1$ such that

$$\inf_{k\ge N_2} \frac{x_{k+1} - x_k}{y_{k+1} - y_k} > c_-.$$

As $y_{k+1} \ge y_k$,

$$x_{k+1} - x_k > c_-(y_{k+1} - y_k), \qquad \forall k \ge N_2.$$

Summing this inequality from $k = N_2$ to $k = n-1$, and simplifying the telescopic sum,

$$x_n - x_{N_2} > c_-(y_n - y_{N_2}), \qquad \forall n > N_2.$$

As $y_n \ne 0$, we can divide by y_n and rearrange,

$$\frac{x_n}{y_n} > c_- + \underbrace{\frac{x_{N_2} - c_- y_{N_2}}{y_n}}_{=:z_n^-}, \qquad \forall n > N_2.$$

By Assumption 1′, $\lim_{n\to\infty} \frac{1}{y_n} = 0$, thus $\lim_{n\to\infty} z_n^- = 0$. So taking liminf of the last inequality and using (11.14), we get

$$c_- \le \varliminf_{n\to\infty} \frac{x_n}{y_n}. \tag{11.15}$$

This holds for every $c_- < \ell$. Taking the supremum over $c_- < \ell$, we get

$$\ell \le \varliminf_{n\to\infty} \frac{x_n}{y_n}.$$

(If $\ell = +\infty$, the "supremum" over $c_- < \ell$ is $\ell = +\infty$.)

Similarly, writing

$$L = \varlimsup_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n},$$

the limsup inequality is automatic when $L = +\infty$, and otherwise, we find that

$$\varlimsup_{n\to\infty} \frac{x_n}{y_n} \le c_+, \qquad \forall\, c_+ > L. \tag{11.16}$$

Taking the infimum over $c_+ > L$, we obtain

$$\varlimsup_{n\to\infty} \frac{x_n}{y_n} \le L.$$

Together with the automatic inequality $\varliminf_{n\to\infty} \frac{x_n}{y_n} \le \varlimsup_{n\to\infty} \frac{x_n}{y_n}$, we have proved (11.13).

Finally, suppose Assumption 3 holds as well, so

$$\ell = L = \lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}.$$

We only spell out the $\ell \in \mathbb{R}$ case. (The $\ell = \pm\infty$ case is left as an exercise.) The inequalities (11.15), (11.16) become

$$c_- < \varliminf_{n\to\infty} \frac{x_n}{y_n} \le \varlimsup_{n\to\infty} \frac{x_n}{y_n} < c_+, \qquad \forall\, c_- < \ell < c_+.$$

(Note that we don't know whether $\lim_{n\to\infty} \frac{x_n}{y_n}$ exists yet.) Taking supremum over $c_- < \ell$ and infimum over $c_+ > \ell$, we get

$$\ell \le \varliminf_{n\to\infty} \frac{x_n}{y_n} \le \varlimsup_{n\to\infty} \frac{x_n}{y_n} \le \ell,$$

so the liminf and limsup of $\frac{x_n}{y_n}$ are equal to ℓ. By Proposition 4.8, the limit exists, and

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \ell.$$

□

Example 11.6 (Example 3.2.3 of Lou–Mei–Yang, slightly modified.). Fix a natural number $k \in \{0, 1, 2, \ldots\}$. Using the Stolz–Cesàro Theorem, we can deduce the following limit,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1^k + 2^k + \ldots + n^k}{n^{k+1}} &= \lim_{n\to\infty} \frac{(n+1)^k}{(n+1)^{k+1} - n^{k+1}} \\
&= \lim_{n\to\infty} \frac{(n+1)^k}{\binom{k+1}{k} n^k + \binom{k+1}{k-1} n^{k-1} + \ldots + \binom{k+1}{1} n + \binom{k+1}{0}} \\
&= \lim_{n\to\infty} \frac{(1+n^{-1})^k}{\binom{k+1}{k} + \binom{k+1}{k-1} n^{-1} + \ldots + \binom{k+1}{1} n^{-(k-1)} + \binom{k+1}{0} n^{-k}} \\
&= \frac{1}{k+1}.
\end{aligned}$$

Exercise 11.2. Consider the sequences

$$x_n = 2n + n\cos(\ln n), \qquad y_n = n, \qquad n \geq 1.$$

The liminf-limsup version of Stolz–Cesàro Theorem is applicable. (Assumptions $1', 2$ only depend on y_n, and are satisfied by $y_n = n$.) But the limit version may not be applicable.

– By considering suitable subsequences, show that

$$\underline{\lim_{n\to\infty}} \cos(\ln n) = -1, \qquad \overline{\lim_{n\to\infty}} \cos(\ln n) = +1. \tag{11.17}$$

– Verify that there are *strict* inequalities

$$\underline{\lim_{n\to\infty}} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} < \underline{\lim_{n\to\infty}} \frac{x_n}{y_n} < \overline{\lim_{n\to\infty}} \frac{x_n}{y_n} < \overline{\lim_{n\to\infty}} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}.$$

Hint: MVT for the function $f(x) = x\cos(\ln x)$ may be helpful for working out

$$\underline{\lim_{n\to\infty}} (x_{n+1} - x_n) \quad \text{and} \quad \overline{\lim_{n\to\infty}} (x_{n+1} - x_n).$$

Corollary 11.3 (Cesáro mean)**.** *For a sequence* $x_1, x_2, \ldots$, *and* $\ell \in \{-\infty\} \sqcup \mathbb{R} \sqcup \{+\infty\}$,

$$\lim_{n\to\infty} x_n = \ell \;\Rightarrow\; \lim_{n\to\infty} \frac{x_1 + x_2 + \ldots + x_n}{n} = \ell.$$

Proof. Set

$$a_n = x_1 + \ldots + x_n, \qquad b_n = n, \qquad n \geq 1.$$

The assumptions of Theorem 11.2 are satisfied by these sequences; e.g.,

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{x_{n+1}}{1} = \ell$$

exists in the extended real numbers. Therefore, we conclude that

$$\ell = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{x_1 + \ldots + x_n}{n}.$$

□

The term $\frac{x_1+\ldots+x_n}{n}$ is called the the n-th *Cesàro mean* of the sequence $x_1, x_2, \ldots$. The converse of Corollary 11.3 is not true!

Exercise 11.3. Consider the sequence $x_n = (-1)^n, n \geq 1$. Show that the its corresponding sequence of Cesàro means converges to 0.

Next, consider the sequence of partial sums,

$$s_n = \sum_{k=1}^{n} x_k.$$

Show that the sequence $\frac{s_1+\ldots+s_n}{n}$ of Cesáro means converges. What is the limit?

If an infinite series $\sum_{k=1}^{\infty} x_k$ has partial sums $s_n = \sum_{k=1}^{n} x_k$ whose Cesáro means converge,

$$\lim_{n\to\infty} \frac{s_1 + \ldots + s_n}{n} \text{ exists in } \mathbb{R},$$

then the series is said to be *Cesàro summable*. Exercise 11.3 shows that it is possible for a series to be non-summable in the usual sense, but *Cesàro summable*.

Exercise 11.4. Let $x_1, x_2, \ldots$ be a sequence of strictly positive real numbers, and let $\ell \in \{-\infty\} \sqcup \mathbb{R} \sqcup \{+\infty\}$.

- Show that

$$\lim_{n\to\infty} x_n = \ell \;\Rightarrow\; \lim_{n\to\infty} (x_1 x_2 \cdots x_n)^{1/n} = \ell.$$

- Show that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell \;\Rightarrow\; \lim_{n\to\infty} x_n^{1/n} = \ell.$$

- There is also a discrete version of the $\frac{0}{0}$ L'Hôpital's Rule at $+\infty$.

Theorem 11.4 (Stolz–Cesàro, $\frac{0}{0}$ version)**.** *Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be real sequences. If*

1. $\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n$*;*
2. $y_1, y_2, \ldots$ *is strictly monotone;*

then

$$\underline{\lim_{n\to\infty}} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} \le \underline{\lim_{n\to\infty}} \frac{x_n}{y_n} \le \overline{\lim_{n\to\infty}} \frac{x_n}{y_n} \le \overline{\lim_{n\to\infty}} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}.$$

Suppose the following also holds,

3. $\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$ *exists in the extended real numbers.*

Then $\lim_{n\to\infty} \frac{x_n}{y_n}$ *exists in the extended real numbers, and equals* $\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$.

Proof. We prove the $\ell = \infty$, strictly monotone decreasing, and limit case. Note that Assumptions 1,2 force $y_n > 0$ for all n (easy proof-by-contradiction).

Let $\mu > 0$ be given. By Assumption 3, there exists $N \ge 1$ such that

$$\frac{x_{n+1}-x_n}{y_{n+1}-y_n} > \mu, \qquad \forall n \ge N.$$

By the strictly monotone decreasing Assumption 2, $y_{n+1} - y_n < 0$, so the above inequality becomes

$$x_{n+1} - x_n < \mu(y_{n+1} - y_n), \qquad \forall n \ge N.$$

By a telescopic sum argument, we get

$$x_m - x_n < \mu(y_m - y_n), \qquad \forall m > n \geq N.$$

Fixing $n \geq N$, taking the $m \to \infty$ limit (Assumption 1), and noting that $y_n > 0$,

$$-x_n \leq -\mu y_n \;\Leftrightarrow\; \frac{x_n}{y_n} \geq \mu, \qquad \forall n \geq N.$$

As $\mu > 0$ was arbitrary, this shows that $\frac{x_n}{y_n}$ tends to $+\infty$ as $n \to \infty$.

The remaining cases and the liminf-limsup version are left as Exercises.

□