2 Lectures 3–4

- There are several ways to introduce the set $\mathbb{R}$ of real numbers. One could just specify the relevant properties $\mathbb{R}$ axiomatically, e.g., "$\mathbb{R}$ is a complete ordered field". This is not very satisfactory, so, to show that such a set exists, there are two main approaches, using *Dedekind cuts*, or *rational Cauchy sequences*. We will take the second approach, and then verify the crucial "completeness" property as a Theorem.

- Exercise (Euclid's division lemma): Let $n \in \mathbb{N}$ and $m \in \mathbb{N} \setminus \{0\}$. Show that there exist $q, r \in \mathbb{N}$ such that

$$n = mq + r, \qquad 0 \leq r < m.$$

  (q is the "quotient" and r is the "remainder".)

  Exercise: Let $x \in \mathbb{Q}$. Show that there exists an integer q such that

$$q \leq x < q + 1.$$

  (Use Euclid's division lemma.)

  So we may picture a rational number x as lying in some "geometric line interval" with endpoints q and $q + 1$. This is only for intuition, since "line interval" is an external concept which we have not defined yet.

- Let us investigate the "gaps" in $\mathbb{Q}$. For this, we introduce the notion of *absolute value* of a rational number $x \in \mathbb{Q}$,

$$|x| := \begin{cases} x, & 0 \leq x, \\ -x, & x \leq 0. \end{cases}$$

  Then $|x - y|$ is the *distance* between $x, y \in \mathbb{Q}$.

  Exercise 2.1. Let $x, y, z \in \mathbb{Q}$. Prove the following statements,

Non-degeneracy. $|x| = 0$ iff $x = 0$. (Thus $|x - y| = 0$ iff $x = y$.)

Triangle inequality. $|x + z| \leq |x| + |z|$. (Thus $|x - z| \leq |x - y| + |y - z|$.)

- $|x| \leq y$ iff $-y \leq x \leq y$.
- $|xy| = |x||y|$.

- Given $x, y \in \mathbb{Q}$ with $x < y$, there exists $z \in \mathbb{Q}$ such that

$$x < z < y.$$

This can be proved using the ordered field property of $\mathbb{Q}$ verified in the exercises. Specifically, take $z = \frac{x+y}{2}$.

Informally, the above result says that "there are no gaps of positive length in $\mathbb{Q}$".

Does this mean that we can label all the points of a geometric line with elements of $\mathbb{Q}$?

- As known since Pythagoras, there is a right-angled triangle with side lengths $1, 1, x$ satisfying

$$x^2 = 1^2 + 1^2 = 2.$$

However, there does *not* exist any $x \in \mathbb{Q}$ satisfying $x^2 = 2$.

Proof-by-contradiction:

  - Suppose $\frac{p}{q} \in \mathbb{Q}$ satisfies $\frac{p^2}{q^2} = 2$. We may represent the rational number $\frac{p}{q}$ in such a way that p, q are not both even integers (why?).
  - We have $p^2 = 2q^2$, thus

$$p \in 2\mathbb{Z} := \{2n \,:\, x \in \mathbb{Z}\}.$$

(Why?)

  - Write $p = 2k$ for suitable integer k. So

$$p^2 = (2k)^2 = 4k^2 = 2q^2.$$

Thus $q^2 = 2k^2$. Same reasoning forces $q \in 2\mathbb{Z}$.

  - This contradicts assumption "p, q are not both even".

This indicates that $\mathbb{Q}$ is insufficient for describing basic geometry — there are line segments that cannot be described just with rational numbers. This problem gets worse when we want to describe circles...

- Fortunately, as we will see, rational numbers are good enough in an *approximate* sense. Roughly speaking, approximation is about controlling differences,

$$\Big|\text{what we have} - \text{what we want}\Big| \leq \text{small number}.$$

Here, we use addition/subtraction and ordering (the fundamental structures on natural/integer/rational numbers) to express approximation. Arguments with such *inequalities* form the heart of the subject of *mathematical analysis*.

As an example, let us find approximate rational solutions to $x^2 = 2$.

Proposition: There exists a sequence $x_1, x_2, \ldots$ of non-negative rational numbers such that

$$x_n^2 < 2 < (x_n + \frac{1}{n})^2, \qquad \forall n \geq 1. \tag{2.1}$$

Proof-by-contradiction: Fix $n \geq 1$. Assume that there does not exist any non-negative rational x satisfying $x^2 < 2 < (x + \frac{1}{n})^2$. By trichotomy, this assumption implies that whenever a non-negative rational x satisfies $x^2 < 2$, it must satisfy $2 \geq (x + \frac{1}{n})^2$. As $2 = (x + \frac{1}{n})^2$ was already shown to be impossible, we have $(x + \frac{1}{n})^2 < 2$. We record this implication below:

$$x \in \mathbb{Q},\ 0 \leq x,\ x^2 < 2 \ \Rightarrow\ (x + \frac{1}{n})^2 < 2. \tag{2.2}$$

- The non-negative rational number 0 satisfies $0^2 < 2$, thus (2.2) implies
$$(0 + \frac{1}{n})^2 < 2.$$
- As $\frac{1}{n}$ is another non-negative rational number, we can apply (2.2) to it, thus
$$(\frac{1}{n} + \frac{1}{n})^2 < 2.$$
- As $\frac{2}{n} + \frac{2}{n} = \frac{2}{n}$ is another non-negative rational number, we can apply (2.2) to it, thus $(\frac{3}{n})^2 < 2$. By induction, we get
$$(\frac{k}{n})^2 < 2, \qquad \forall k \in \mathbb{N}.$$

- Since n, thus $2n^2$, was fixed, there exists some large $k \in \mathbb{N}$ such that $k^2 > 2n^2$. This contradicts the previous inequality.
- Our assumption must therefore be false — there *does* exist some non-negative rational number x_n satisfying $x_n^2 < 2 < (x_n + \frac{1}{n})^2$.
- Applying this proof-by-contradiction for each natural number $n \geq 1$, gives the desired sequence of rational numbers satisfying (2.1).

- It is quite easy to show that for non-negative rational numbers a, b,

$$a^2 < b^2 \;\Rightarrow\; a < b.$$

So, for the sequence $x_1, x_2, \ldots, x_n, \ldots$ of non-negative rational numbers provided by the above Proposition, the inequalities (2.1) imply that

$$x_n < \text{“}\sqrt{2}\text{”} < x_n + \frac{1}{n}, \qquad n \geq 1.$$

There are quotation marks around the middle term “$\sqrt{2}$”, because we have not defined $\sqrt{2}$ yet. Later, when real numbers have been defined, “$\sqrt{2}$” will be a real number which lies between x_n and $x_n + \frac{1}{n}$ *for every* $n \geq 1$.

The distance between x_n and “$\sqrt{2}$” is smaller than $\frac{1}{n} = |x_n - (x_n + \frac{1}{n})|$, so it gets smaller as n is chosen to be larger. For example, we can use x_{10000} as a rational approximation to “$\sqrt{2}$”, with an error of at most $\frac{1}{10000}$.

- We will define the “real numbers” using *sequences* of rational numbers with the same kind of “convergent” behaviour as the sequence from the Proposition.

More precisely, we consider *rational Cauchy sequences*, defined to be sequences $x_1, x_2, \ldots, x_n, \ldots$ of rational numbers, satisfying:

(ℚ) **Cauchy property**. Given any rational $\epsilon > 0$, there exists some natural number N such that

$$|x_m - x_n| < \epsilon, \qquad \forall\, m, n \geq N.$$

- Notational remark: A sequence in a set X is a function $f : \mathbb{N} \to X$. For convenience, we will generally start the sequence from the label 1 instead of 0, so actually it is a function $f : \mathbb{N} \setminus \{0\} \to X$. We usually just say that “$x_1, x_2, \ldots$ is a sequence in X”.

To specify a particular sequence, we may provide a formula x_n for the general n-th term, $n \geq 1$. For example, the sequence

$$-1, 1, -1, +1, \ldots, (-1)^n, \ldots$$

has n-th term being $(-1)^n$. This sequence is not Cauchy. This is because $|x_n - x_{n-1}| = 2$ for all n, so given $\epsilon = 1$, there does *not* exist N satisfying $|x_m - x_n| < 1$ for all $m, n \geq N$.

A constant sequence $x, x, \ldots, x, \ldots$, with $x \in \mathbb{Q}$, is obviously a rational Cauchy sequence.

Exercise: Show that the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ satisfies the Cauchy property.

Exercise 2.2. Show that a rational Cauchy sequence must be bounded, i.e., $\exists M \in \mathbb{Q}$ such that $|x_n| \leq M$ for all $n \geq 1$.

- Let $x_1, x_2, \ldots, x_n, \ldots$ be a sequence of non-negative rational numbers satisfying (2.1), which is available because of the above Proposition. Let us prove that it satisfies the Cauchy property.

  First, we observe that

  $$\forall n \geq 3, \qquad x_n \geq 1.$$

  (Otherwise, if $x_n < 1$, then $(x_n + \frac{1}{n})^2 < (1 + \frac{1}{n})^3 \leq (1 + \frac{1}{3})^2 = \frac{16}{9} < 2$, contradicting (2.1).

  So, if we fix $N \geq 3$, then for all $m, n \geq N$, we have

$$\begin{aligned}
|x_m - x_n| &= |x_m^2 - x_n^2|/|x_m + x_n| && (x_m > 0, x_n > 0)\\
&= |(x_m^2 - 2) + (2 - x_n^2)|/(x_m + x_n)\\
&\leq \big(|x_m^2 - 2| + |2 - x_n^2|\big)/(x_m + x_n) && \text{(triangle inequality)}\\
&= \big((2 - x_m^2) + (2 - x_n^2)\big)/(x_m + x_n) && \text{(Equation (2.1))}\\
&< \Big(\big(x_m + \frac{1}{m}\big)^2 - x_m^2 + \big(x_n + \frac{1}{n}\big)^2 - x_n^2\Big)/(x_m + x_n) && \text{(Equation (2.1))}\\
&= \Big(\frac{2x_m}{m} + \frac{1}{m^2} + \frac{2x_n}{n} + \frac{1}{n^2}\Big)/(x_m + x_n)\\
&\leq \frac{2(x_m + x_n)}{N(x_m + x_n)} + \frac{2}{N^2(x_m + x_n)} && (\frac{1}{m}, \frac{1}{n} \leq \frac{1}{N})\\
&\leq \frac{2}{N} + \frac{1}{N^2}. && (\frac{1}{x_m + x_n} \leq \frac{1}{1+1} = \frac{1}{2})
\end{aligned} \tag{2.3}$$

Now, when we are given a positive rational $\epsilon > 0$, we simply choose N to be a natural number satisfying

$$N \geq 3 \quad \text{and} \quad N \geq \frac{3}{\epsilon} \ (\Rightarrow N^2 \geq N \geq \frac{3}{\epsilon})$$

(Such an N exists by an exercise). Putting this choice of N into the inequality (2.3), we get

$$\forall m, n \geq N, \qquad |x_m - x_n| < \frac{2}{3/\epsilon} + \frac{1}{3/\epsilon} = \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

verifying that the Cauchy property holds. □

Remark: If the given $\epsilon > 0$ is very small, we have to choose a very large N in the above proof. No single N will work for every $\epsilon > 0$. To emphasize that the choice of N depends on the specific given $\epsilon > 0$, we sometimes write $N \equiv N(\epsilon)$.

- We would like all choices of rational Cauchy sequences provided by the Proposition to represent the *same* "limiting number $\sqrt{2}$". So we should collapse all such choices into a single equivalence class. The appropriate equivalence relation is given below.

  We say that two rational Cauchy sequences,

  $$a_1, a_2, \ldots, \quad \text{and} \quad b_1, b_2, \ldots,$$

  are *equivalent*, if for any rational $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

  $$|a_n - b_n| < \epsilon, \qquad \forall n \geq N. \tag{2.4}$$

  (After the $N \equiv N(\epsilon)$-th term, the two sequences coincide up to an error of at most ϵ.)

  It is quite easy to verify the equivalence relation property:

  - Reflexivity is automatic, and symmetry is obvious since the condition (2.4) is unchanged when a_n and b_n are swapped.
  - For transitivity, suppose $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are equivalent rational Cauchy sequences, and that $b_1, b_2, \ldots$ is also equivalent to

$c_1, c_2, \ldots$. So, given $\epsilon > 0$, thus $\epsilon/2 > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} |a_n - b_n| &< \epsilon/2, && \forall n \geq N_1, \\ |b_n - c_n| &< \epsilon/2, && \forall n \geq N_2. \end{aligned}$$

Set $N = \max\{N_1, N_2\}$. By the triangle inequality,

$$\begin{aligned} |a_n - c_n| &= |(a_n - b_n) + (b_n - c_n)| \\ &\leq |a_n - b_n| + |b_n - c_n| \\ &< \epsilon/2 + \epsilon/2 = \epsilon, && \forall n \geq N, \end{aligned}$$

verifying that $a_1, a_2, \ldots$ and $c_1, c_2, \ldots$ are equivalent.

- The Cauchy property depends only on the "tail" of a sequence: we can freely modify finitely many terms (e.g., the first M terms), without affecting the Cauchy property. We just need to adjust the choice of $N \equiv N(\epsilon)$ to be larger than M, when checking the Cauchy condition.

  Likewise, the equivalence relation for Cauchy sequences depends only on the "tails" of the Cauchy sequences.

- Intuitively, the Cauchy property expresses the idea that a Cauchy sequence "converges" to some "limiting number". The quotation marks indicate that we have not defined those words yet.

  Definition 1. The set $\mathbb{R}$ of *real numbers* is the set of equivalence classes of rational Cauchy sequences.

  – A constant rational sequence with $x_n = x \in \mathbb{Q}$ for all $n \geq 1$, is certainly a rational Cauchy sequence. So its equivalence class is a special case of a real number, called a *rational real number*.

  Note that a constant rational sequence $x, x, x, \ldots$ can be equivalent to a non-constant sequence, for example,

  $$x + 1, x + \frac{1}{2}, x + \frac{1}{3}, \ldots, x + \frac{1}{n}, \ldots$$

  (Exercise.)

– Suppose $x, y \in \mathbb{Q}$ are distinct, then the constant Cauchy sequences

$$x, x, x, \ldots \qquad \text{and} \qquad y, y, y, \ldots$$

are inequivalent. This is because if we are given a rational $\epsilon = \frac{|x-y|}{2} > 0$, no matter which position we start from, the n-th terms continue to differ by $|x - y| > \epsilon$. So the Cauchy property cannot hold.

Therefore, there is an *injective* map

$$\begin{aligned} \iota : \mathbb{Q} &\to \mathbb{R} \\ x &\mapsto [x, x, x, \ldots]. \end{aligned}$$

When a rational $x \in \mathbb{Q}$ is regarded as a real number, it is customary to just write $x \in \mathbb{R}$ instead of $\iota(x) = [x, x, x, \ldots] \in \mathbb{R}$.

- Let us extend the notions of addition and multiplication on $\mathbb{Q}$ to the larger set $\mathbb{R}$.

  We define the sum of two rational sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, to be

  $$(a_1, a_2, \ldots) + (b_1, b_2, \ldots) := a_1 + b_1, a_2 + b_2, \ldots,$$

  and their product to be the sequence

  $$(a_1, a_2, \ldots) \times (b_1, b_2, \ldots) := a_1 b_1, a_2 b_2, \ldots.$$

  Suppose the rational sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are *Cauchy*. Let $\epsilon > 0$ be given. The Cauchy property means that there exist $N_a, N_b \in \mathbb{N}$ such that

  $$\begin{aligned} |a_m - a_n| &< \epsilon/2, && \forall m, n \geq N_a, \\ |b_m - b_n| &< \epsilon/2, && \forall m, n \geq N_b. \end{aligned}$$

  Set $N = \max\{N_a, N_b\}$. Then the triangle inequality ensures that

  $$\begin{aligned} |(a_m + b_m) - (a_n + b_n)| &= |(a_m - a_n) + (b_m - b_n)| \\ &\leq |a_m - a_n| + |b_m - b_n| \\ &< \epsilon/2 + \epsilon/2 = \epsilon, && \forall m, n \geq N. \end{aligned}$$

This verifies that the sum of two rational *Cauchy* sequences is another (rational) *Cauchy* sequence.

Exercise: Show that the product of two rational Cauchy sequences is another rational Cauchy sequence.

Suppose $b'_1, b'_2, \ldots$ and $b_1, b_2, \ldots$ are equivalent rational Cauchy sequences. Their n-th terms differ by $|b_n - b'_n|$. Let us add another rational Cauchy sequence $a_1, a_2, \ldots$ to each of them. The resulting Cauchy sequences,

$$(a_1, a_2, \ldots) + (b_1, b_2, \ldots) \quad \text{and} \quad (a_1, a_2, \ldots) + (b'_1, b'_2, \ldots), \tag{2.5}$$

still differ at the n-th term by the same number $|b_n - b'_n|$. It follows that the sequences in (2.5) are also equivalent. Similarly if the roles of a_n and b_n are swapped. It follows that the sum of *equivalence classes* of rational Cauchy sequences (i.e., the sum of real numbers) is well-defined,

$$[a_1, a_2, \ldots] + [b_1, b_2, \ldots] := [a_1 + b_1, a_2 + b_2, \ldots]. \tag{2.6}$$

Exercise: With a_n, b_n, b'_n as above, show that the product sequences

$$a_1 b_1, a_2 b_2, \ldots \quad \text{and} \quad a_1 b'_1, a_2 b'_2, \ldots$$

are still equivalent.

From these exercises, it follows that the product of *equivalence classes* of rational Cauchy sequences (i.e., real numbers) is well-defined,

$$[a_1, a_2, \ldots] \times [b_1, b_2, \ldots] := [a_1 b_1, a_2 b_2, \ldots]. \tag{2.7}$$

- Exercise: Let $x_1, x_2, \ldots$ be a rational Cauchy sequence approximating "$\sqrt{2}$", as provided by the Proposition. Show that
  - All such approximating sequences are equivalent.
    So it makes sense to define the real number
    $$\sqrt{2} := [x_1, x_2, \ldots] \in \mathbb{R}$$
    as the equivalence class of any such rational approximating sequence.
  - No such approximating sequence is equivalent to a constant rational sequence. (Thus $\sqrt{2}$ is not a rational real number.)

The complement of the rational numbers,

$$\mathbb{R} \setminus \iota(\mathbb{Q}),$$

is the set of *irrational* real numbers. It is non-empty since it contains $\sqrt{2}$.

Notation: Often, we will just write a real number as $x \in \mathbb{R}$, instead of presenting it as an equivalence class of rational Cauchy sequences.

- We record the familiar algebraic properties of $(\mathbb{R}, +, \times)$; they are the same as those of $(\mathbb{Q}, +, \times)$
  - Addition is commutative and associative.
  - The zero element is $0 \equiv [0, 0, \ldots]$, and the additive inverse of $x \equiv [x_1, x_2, \ldots]$ is
    $$-x = [-x_1, -x_2, \ldots].$$
    Subtraction $x - y$ means $x + (-y)$, as usual.
  - Multiplication is commutative and associative.
  - The distributive law holds.
  - The multiplicative identity is $1 \equiv [1, 1, \ldots]$, and every nonzero $x \in \mathbb{R}$ has an (unique) inverse $x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$.
    Division x/y means $x \times y^{-1}$, as usual.

  These properties are summarized as: "$(\mathbb{R}, +, \times)$ is a *field*".

  We only prove the existence of multiplicative inverses, which is quite tricky. (You may wish to prove the other algebraic properties by yourself.)

- Let $x_1, x_2, \ldots$ be a rational Cauchy sequence representing a *non-zero* real number $x \in \mathbb{R}$. It is inequivalent to the constant sequence $0, 0, \ldots$, so there must exist some rational $\epsilon > 0$ such that

  $$\forall N \geq 1,\ \exists n_N \geq N \ \text{ such that } \ |x_{n_N}| \geq \epsilon. \tag{2.8}$$

  Since $x_1, x_2, \ldots$ is Cauchy, given the above rational $\epsilon > 0$, there exists a natural number M such that

  $$|x_m - x_n| < \epsilon/2, \qquad \forall m, n \geq M. \tag{2.9}$$

Setting $N = M$ in (2.8), we obtain a term x_{n_M} of the sequence that satisfies

$$|x_{n_M}| \geq \epsilon. \qquad n_M \geq M. \tag{2.10}$$

Combining the triangle inequality with the inequalities (2.9) and (2.10), we have

$$\begin{aligned} \forall n \geq M, \qquad \epsilon \overbrace{\leq}^{(2.10)} |x_{n_M}| &\leq |x_{n_M} - x_n| + |x_n| \overbrace{<}^{(2.9)} \epsilon/2 + |x_n| \\ &\Rightarrow \epsilon/2 < |x_n|. \end{aligned}$$

Now the original sequence $x_1, x_2, \ldots, x_{M-1}, x_M, x_{M+1}, \ldots$ is equivalent to

$$x_M, x_M, \ldots, x_M, x_M, x_{M+1}, \ldots, \tag{2.11}$$

since we have only modified the first $M-1$ terms. So the modified rational Cauchy sequence (2.11) represents the same real number x, but it has the advantage that *all* of its terms satisfy $|x_n| \geq \epsilon/2 > 0$.

(Without making the modification, some of the terms might be 0 and cannot be inverted.)

- The above construction tells us that a *nonzero* $x \in \mathbb{R}$ is representable by a rational Cauchy sequence $x_1, x_2, \ldots$ with the property that

$$\text{for some rational } c > 0, \qquad |x_n| \geq c, \qquad \forall n \geq 1. \tag{2.12}$$

We can then consider the rational sequence of reciprocals,

$$x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}, \ldots. \tag{2.13}$$

Exercise: Show that the reciprocal rational sequence (2.13) is Cauchy.

Let $\tilde{x}_1, \tilde{x}_2, \ldots$ be some other rational Cauchy sequence representing $x \in \mathbb{R}$, with the property that there exists some $\tilde{c} > 0$ such that $|\tilde{x}_n| \geq c$ for all $n \geq 1$. According to the well-defined rule of multiplying equivalence classes of rational Cauchy sequences,

$$\begin{aligned} [x_1^{-1}, x_2^{-1}, \ldots] &= [x_1^{-1}, x_2^{-1}, \ldots] \times [\tilde{x}_1 \tilde{x}_1^{-1}, \tilde{x}_2 \tilde{x}_2^{-1}, \ldots] \\ &= [x_1^{-1}, x_2^{-1}, \ldots] \times [\tilde{x}_1, \tilde{x}_2, \ldots] \times [\tilde{x}_1^{-1}, \tilde{x}_2^{-1}, \ldots] \\ &= [x_1^{-1}, x_2^{-1}, \ldots] \times [x_1, x_2, \ldots] \times [\tilde{x}_1^{-1}, \tilde{x}_2^{-1}, \ldots] \\ &= [x_1^{-1} x_1, x_2^{-1} x_1, \ldots] \times [\tilde{x}_1^{-1}, \tilde{x}_2^{-1}, \ldots] \\ &= [\tilde{x}_1^{-1}, \tilde{x}_2^{-1}, \ldots]. \end{aligned}$$

So equivalent choices of $x_1, x_2, \ldots$ representing x (and satisfying (2.12)) lead to equivalent reciprocal sequences.

Thus, for nonzero $x \in \mathbb{R}$,

$$x^{-1} := [x_1^{-1}, x_2^{-1}, \ldots]$$

is a well-defined real number, such that $xx^{-1} = 1$ holds. This verifies that every nonzero real number has a well-defined multiplicative inverse in $\mathbb{R}$.

- For analysis, it is essential to have an ordering on $\mathbb{R}$.

  We define a real number $x \in \mathbb{R}$ to be *positive*, if it can be represented by a rational Cauchy sequence $x_1, x_2, \ldots$ such that for some positive rational c,

  $$x_n \geq c, \qquad \forall n \geq 1, \tag{2.14}$$

  holds. We also say that $x \in \mathbb{R}$ is *negative* if $-x$ is positive.

  Check that for rational real numbers, this notion of positive/negative is the same as the previously-defined one.

  Exercise: Let x be a non-zero real number. We saw that x is representable by a rational Cauchy sequence satisfying (2.12) for some positive $c > 0$. Deduce that either x or $-x$ must be positive.

  Thus, there is exactly one possibility,

  $$\text{negative},\ 0,\ \text{positive},$$

  for each real number.

  Now we can define the ordering on $\mathbb{R}$, extending the ordering on $\mathbb{Q}$:

  $$x \leq y \quad \text{iff} \quad y - x \text{ is positive, or } x = y.$$

  We omit the proofs of the following facts on the ordering $\leq$ on $\mathbb{R}$:

  - $\leq$ is a total order.
  - For $x, y \in \mathbb{R}$, exactly one of

    $$x < y, \qquad x = y, \qquad y < x$$

    is true (trichotomy).

– The relation $\leq$ is preserved by addition.

– The relation $<$ is preserved by multiplication by positive real numbers. Thus

$$0 \leq x \text{ and } 0 \leq y \Rightarrow 0 \leq xy.$$

To summarize: Just like the rational numbers, $(\mathbb{R}, +, \times, \leq)$ is an ordered field.