3 Lectures 5–6

- The following is a "geometrically obvious" property of "the real line". But given only the algebraic construction of $(\mathbb{R}, +, \times, \leq)$, it actually requires some work to verify.

  Proposition 3.1 (Archimedean property)**.** *Let $x \in \mathbb{R}$. Given any positive real number $\epsilon > 0$, there exists a positive integer M such that $M\epsilon > x$.*

  Proof. Suppose $x \in \mathbb{R}$ with $x \leq 0$. Given a real $\epsilon > 0$, just choose $M = 1$ to satisfy the Archimedean property.

  Otherwise, suppose $x \in \mathbb{R}$ is positive, $x > 0$. To proceed, we first prove a few intermediate claims.

  - First Claim: If $a_1, a_2, \ldots$ is a rational Cauchy sequence whose terms are non-negative, then $a = [a_1, a_2, \ldots]$ is a non-negative real number.

    Suppose otherwise, that $a < 0$. By definition (see (2.14)), we can write

    $$a = [b_1, b_2, \ldots]$$

    where $b_1, b_2, \ldots$ is some rational Cauchy sequence that satisfies $b_n \leq -c < 0$ for some rational $-c < 0$. Then

    $$|a_n - b_n| = a_n - b_n \geq a_n - (-c) = a_n + c \geq c \qquad \forall n \geq 1,$$

    so the sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are inequivalent. However, they represent the same real number a, so there is a contradiction.

  - Second Claim: Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be rational Cauchy sequences representing real numbers a and b respectively. If $a_n \leq b_n$ for all $n \geq 1$, then $a \leq b$.

    This follows by applying the first claim to the non-negative sequence $b_1 - a_1, b_2 - a_2, \ldots$.

  - Third Claim: Let $a > 0$ be a positive real number, then there exists a positive integer N such that

    $$a \leq N.$$

Proof of claim: By definition of $a > 0$, there exists a rational $c > 0$ and some rational Cauchy sequence $a_1, a_2, \ldots$, with $a_n \geq c > 0$ for all $n \geq 1$. In an exercise, we showed that rational Cauchy sequences are bounded, so there exists some (positive) $M \in \mathbb{Q}$ such that

$$0 < c \leq a_n \leq M, \qquad \forall n \geq 1.$$

By the exercise after Euclid's division lemma, there exists some positive integer N with $M \leq N$. For this N, we have

$$a_n \leq N, \qquad \forall n \geq 1.$$

Regarding N as the real number represented by the constant sequence $N, N, \ldots$, we conclude from the Second Claim that $a \leq N$.

Return to the original problem. We are studying the remaining case where $x > 0$ is a positive real number. Let $\epsilon > 0$ be a given positive real number. Since $\frac{x}{\epsilon} > 0$, we can use the Third Claim to obtain some positive integer N such that $\frac{x}{\epsilon} \leq N$. Set $M = N + 1$ to verify the Archimedean property $x < M\epsilon$. □

Corollary 3.2. *If x, y are real numbers with $x < y$, then there exists a rational number p such that*

$$x < p < y. \tag{3.1}$$

Proof. First, for a positive real number x, Use Proposition 3.1 to show that there exists a positive integer N such that $0 < \frac{1}{N} < x$. The rest of the proof is also left as an Exercise. □

Corollary 3.3. *Let $x_1, x_2, \ldots$ be a rational Cauchy sequence representing a real number x. Let y be a real number such that*

$$x_n \leq y, \qquad \forall n \geq 1,$$

holds. Then $x \leq y$ holds.

Proof-by-contradiction. Suppose $y < x$. Then we can use Corollary 3.2 to find a rational p with

$$y < p < x.$$

By assumption, $x_n \leq y$ for all $n \geq 1$, so in combination with (3.1), we have $x_n < p$ for all $n \geq 1$. Apply the Second Claim in the proof of Archimedean property to get $x \leq p$. This contradicts $p < x$ in (3.1). □

- The $\mathbb{Q}$-valued absolute value function on $\mathbb{Q}$ extends to an $\mathbb{R}$-valued function on $\mathbb{R}$ (with the same formula). Likewise, we obtain a *real-valued distance* between a pair of real numbers,

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (x, y) &\mapsto |x - y|, \end{aligned} \tag{3.2}$$

  extending the $\mathbb{Q}$-valued distance between rational numbers.

  The properties of $|\cdot|, \leq$ *listed in Exercise 2.1 remain valid over* $\mathbb{R}$. (The proof is left as an exercise.)

  The following statement is usually summarized as "$\mathbb{Q}$ is a dense subset of $\mathbb{R}$", meaning that each real number is arbitrarily near to some rational number.

  Proposition 3.4 (Density)**.** *Let x be a real number. Given any real $\epsilon > 0$, there exists a rational real number p such that $|x - p| < \epsilon$.*

  Proof. By Corollary 3.2, there exists a rational p such that

$$x < p < x + \epsilon.$$

  Thus

$$|x - p| = |p - x| = p - x < \epsilon.$$

  □

- (Geometric remark.) The conceptual roles of $\mathbb{R}$ on the two sides of (3.2) are different. In more advanced courses, one replaces the $\mathbb{R}$ on the left side of (3.2) by some general set X, then a real-valued distance function[1] $d : X \times X \to \mathbb{R}$ allows X to be studied as a *geometric* space, technically called a *metric space*.

  Once we know what distances between points on X mean, we can talk about Cauchy sequences in X, in the same way as we did for the case $X = \mathbb{Q}$. So the concept of Cauchy sequences is a special case of a general *geometric* concept. Below, we write out the Cauchy condition for sequences in $X = \mathbb{R}$.

[1] A distance function on X satisfies $d(x, x) = 0$, the triangle inequality, etc., much like the distance function (3.2) on $\mathbb{R} \times \mathbb{R}$.

- A sequence $x_1, x_2, \ldots$ of *real* numbers is said to be a *Cauchy sequence*, if it satisfies:

($\mathbb{R}$) **Cauchy property**. Given any *real* $\epsilon > 0$, there exists some natural number N such that

$$|x_m - x_n| < \epsilon, \qquad \forall\, m, n \geq N.$$

(Note: each real number x_n in the sequence is itself an equivalence class of rational Cauchy sequences.)

The same proof as in Exercise 2.2 shows that any $\mathbb{R}$-Cauchy sequence is bounded by some real number $M \in \mathbb{R}$.

- A rational sequence $x_1, x_2, \ldots$ may be regarded as a special case of a *real* number sequence, whose terms x_n and distances $|x_m - x_n|$ just happen to be rational real numbers. In this situation, there are two notions of Cauchy sequence: the $\mathbb{Q}$-Cauchy one, and the $\mathbb{R}$-Cauchy one. These two notions coincide, so there is no ambiguity:
  - A given real $\epsilon > 0$ is also a rational $\epsilon > 0$, so the $\mathbb{R}$-Cauchy property implies the $\mathbb{Q}$-Cauchy property.
  - Conversely, suppose a rational sequence is $\mathbb{Q}$-Cauchy. Then given a real $\epsilon > 0$, use Corollary 3.2 to find a rational $\epsilon' > 0$ with $\epsilon' < \epsilon$. The $\mathbb{Q}$-Cauchy property gives an N such that $|x_m - x_n| < \epsilon' < \epsilon$ for all $m, n \geq N$. So the $\mathbb{R}$-Cauchy property is also satisfied.

  Now, we know that a real number $x \in \mathbb{R}$ is represented by some Cauchy sequence $x_1, x_2, \ldots$ of rational numbers. We can regard such a rational sequence as a *real* sequence, and we expect that this real sequence "converges to the real number x". This expectation turns out to be true (Lemma 3.5), but we need to first define precisely what "converges to the real number x" means.

- The following definition is fundamental to the subject of mathematical analysis.

  Definition 2. A sequence $x_1, x_2, \ldots$ of real numbers is *convergent*, with *limit* $L \in \mathbb{R}$, if

  > Given any positive real number $\epsilon > 0$, there exists a natural number N such that
  >
  > $$|x_n - L| < \epsilon, \qquad \forall n \geq N.$$

In this case, we write

$$\lim_{n\to\infty} x_n = L.$$

Otherwise, we say that the sequence is *divergent*.

Note that N generally depends on the given ϵ, so we sometimes write $N \equiv N(\epsilon)$ to emphasize this dependence.

Like the Cauchy property, the convergence property of a sequence depends only on its tail.

Exercise 3.1. Show that the sequences

$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$$

and

$$\frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, \ldots$$

both converge to the limit $L = 0$, in the sense of Definition 2.

Example 3.1. The sequence whose n-th term is $\frac{n}{2n+1}$ is convergent. Indeed,

$$\frac{n}{2n+1} = \frac{n + \frac{1}{2} - \frac{1}{2}}{2n+1} = \frac{1}{2} - \frac{1}{2(2n+1)}.$$

So, given $\epsilon > 0$, pick $N = \frac{1}{4\epsilon}$, so that

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| = \left|\frac{1}{2(2n+1)}\right| < \frac{1}{4n} \leq \frac{1}{4N} = \epsilon, \qquad \forall n \geq N.$$

This shows that $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$.

There are many variations of sequences like the one in Example 3.1, and it would be very inefficient to find explicit choices of $N \equiv N(\epsilon)$ for every individual example sequence. It would be better to have some rules for reasoning with limits.

- Here are some basic properties of limits.

  Exercise 3.2 (Uniqueness of limit). Using the triangle inequality, verify that if a sequence of real numbers converges to $L \in \mathbb{R}$ as well as $L' \in \mathbb{R}$, then $L = L'$.

Exercise 3.3 (Preservation of monotonicity). Let $x_1 \geq x_2 \geq \ldots \geq x_n \geq \ldots$ be a non-increasing sequence of real numbers. Suppose $\lim_{n\to\infty} = x$. Show that $x_n \geq x$ for all $n \geq 1$.

Exercise 3.4 (Sum, product, and quotient rules for limits). Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be convergent sequences of real numbers, with respective limits $a, b \in \mathbb{R}$, i.e.,

$$\lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} b_n = b.$$

– Show that

$$\lim_{n\to\infty} (a_n \pm b_n) = a \pm b,$$
$$\lim_{n\to\infty} a_n b_n = ab.$$

– If $b \neq 0$ and $b_n \neq 0$ for all $n \geq 1$, show that

$$\lim_{n\to\infty} b_n^{-1} = b^{-1}$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Using the above algebraic rules, we can more directly deduce, e.g., the limit in Example 3.1 as follows,

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{2+\lim_{n\to\infty} \frac{1}{n}} = \frac{1}{2+0} = \frac{1}{2}.$$

Exercise 3.5 (Subsequence). Given a sequence $x_1, x_2, \ldots, x_n, \ldots$, a *subsequence* is a sequence $y_1, y_2, \ldots, y_k, \ldots$, where

$$y_k = x_{n_k}, \qquad \text{with } \; n_k < n_{k+1} \;\; \forall k \geq 1.$$

For example, the *even subsequence* is

$$x_2, x_4, \ldots, x_{2k}, \ldots, \qquad k \geq 1,$$

while the *odd* subsequence is

$$x_1, x_3, \ldots, x_{2k-1}, \ldots, \qquad k \geq 1.$$

Show that:

– If a sequence of real numbers converges to $L \in \mathbb{R}$, then all of its subsequences also converge to L.
– A sequence of real numbers converges to $L \in \mathbb{R}$ iff its even and odd subsequences converge to $L \in \mathbb{R}$.

Exercise 3.6 (Squeezing principle). Let $a_1, a_2, \ldots$ and $c_1, c_2, \ldots$ be sequences of real numbers, such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n.$$

– Show that if $b_1, b_2, \ldots$ is a sequence of real numbers such that

$$a_n \leq b_n \leq c_n, \qquad \forall n \geq 1,$$

then $b_1, b_2, \ldots$ is also convergent, with

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

– Deduce that if $x_1, x_2, \ldots$ is a sequence of real numbers such that

$$\lim_{n\to\infty} |x_n| = 0,$$

then

$$\lim_{n\to\infty} x_n = 0.$$

Exercise 3.7. We will explain the trigonometric function sin in a later lecture. At the moment, we only need to know that $-1 \leq \sin(x) \leq 1$ for all $x \in \mathbb{R}$. Using the squeezing principle, show that the sequence whose n-th term is

$$x_n = \frac{\sin n}{n}$$

converges to 0.

Lemma 3.5. *A rational Cauchy sequence $x_1, x_2, \ldots$ converges to $[x_1, x_2, \ldots] \in \mathbb{R}$.*

Proof-by-contradiction. Suppose $x_1, x_2, \ldots$ does *not* converge to $x = [x_1, x_2, \ldots]$. Then there exists some real $\epsilon > 0$, such that for all $N \geq 1$,

$$\exists\, n_N \geq N \text{ such that } |x_{n_N} - x| \geq \epsilon. \tag{3.3}$$

For the above ϵ, the Cauchy property provides a natural number M such that

$$|x_m - x_n| < \epsilon/2, \qquad \forall m, n \geq M. \tag{3.4}$$

Set $N = M$ in (3.3), to get

$$|x_{n_M} - x| \geq \epsilon \;\Rightarrow\; \begin{cases} x_{n_M} - x \geq \epsilon, & \text{or} \\ x - x_{n_M} \geq \epsilon. \end{cases} \tag{3.5}$$

As $n_M \geq M$ (by its definition in Eq. (3.3)), Eq. (3.4) also implies

$$|x_m - x_{n_M}| < \epsilon/2, \qquad \forall m \geq n_M. \tag{3.6}$$

In the first case of (3.5), we have

$$x_m - x = (\underbrace{x_m - x_{n_M}}_{>-\epsilon/2 \text{ by } (3.6)}) + (\underbrace{x_{n_M} - x}_{\geq \epsilon}) > \epsilon/2, \qquad \forall m \geq n_M.$$

In the second case of (3.5), we have

$$x - x_m = (\underbrace{x - x_{n_M}}_{\geq \epsilon}) + (\underbrace{x_{n_M} - x_m}_{>-\epsilon/2 \text{ by } (3.6)}) > \epsilon/2, \qquad \forall m \geq n_M.$$

So

$$\begin{cases} x + \epsilon/2 < x_m, & \forall m \geq n_M, \quad \text{or} \\ x - \epsilon/2 > x_m, & \forall m \geq n_M. \end{cases}$$

By Corollary 3.3, and the fact that the real number

$$x = [x_1, x_2, \ldots, x_{n_M - 1}, x_{n_M}, x_{n_M + 1}, \ldots]$$

depends only on the tail of the representative Cauchy sequence, we obtain

$$\begin{cases} x + \epsilon/2 \leq x, & \text{or} \\ x - \epsilon/2 \geq x, \end{cases}$$

which is a contradiction in either case. □

- Suppose a sequence $x_1, x_2, \ldots$ of real numbers converges to $L \in \mathbb{R}$. So when a real $\epsilon > 0$ is given, we can find a natural number N such that

$$|x_n - L| < \epsilon/2, \qquad \forall n \geq N.$$

Then, by the triangle inequality,

$$|x_n - x_m| \leq |x_n - L| + |L - x_m| < \epsilon/2 + \epsilon/2 = \epsilon, \qquad \forall m, n \geq N,$$

which is the Cauchy property. We have thus proven the following.

Lemma 3.6. *A convergent sequence of real numbers is a Cauchy sequence.*

Exercise 3.8. The *harmonic series* is the sequence $S_1, S_2, \ldots$ whose n-th term is the sum[2]

$$S_n = \sum_{1 \le k \le n} \frac{1}{k} := 1 + \frac{1}{2} + \ldots + \frac{1}{n}.$$

Show that the harmonic series is *not* a Cauchy sequence.

(Thus it is divergent by Lemma 3.6.)

Proposition 3.7 (Cauchy completeness/criterion)**.** *A sequence of real numbers is convergent iff it is Cauchy.*

Proof. By Lemma 3.6, we only need to show the $\Leftarrow$ statement: any Cauchy sequence of real numbers is convergent.

So, let $x_1, x_2, \ldots$ be a Cauchy sequence of real numbers. For each $k \ge 1$, apply the density Proposition 3.4 to the real number x_k to obtain rational numbers $x_{k,n}$ with the property that

$$|x_k - x_{k,n}| < \frac{1}{n}, \qquad \forall n \ge 1, \forall k \ge 1.$$

Let a real $\epsilon > 0$ be given. Fix a natural number N such that $N > 3/\epsilon$ (Archimedean property), then

$$|x_k - x_{k,n}| < \frac{1}{N} < \epsilon/3, \qquad \forall n \ge N, k \ge 1. \tag{3.7}$$

Since $x_1, x_2, \ldots$ is assumed to be Cauchy, there exists another natural number N' such that

$$|x_k - x_l| < \epsilon/3, \qquad \forall k, l \ge N'. \tag{3.8}$$

So, setting $M = \max\{N, N'\}$, we obtain from (3.7)–(3.8)

$$\begin{cases} |x_k - x_{k,k}| < \epsilon/3, \\ |x_l - x_{l,l}| < \epsilon/3 \\ |x_k - x_l| < \epsilon/3, \end{cases} \qquad \forall k, l \ge M. \tag{3.9}$$

[2]Strictly speaking, the sums $S_n, n \ge 1$, are recursively defined, $S_{n+1} = S_n + \frac{1}{n+1}$.

Using the triangle inequality and (3.9), we have

$$\begin{aligned} |x_{k,k} - x_{l,l}| &\leq |x_{k,k} - x_k| + |x_k - x_l| + |x_l - x_{l,l}| \\ &< \epsilon, \qquad\qquad \forall k, l \geq M. \end{aligned}$$

This shows that $x_{1,1}, x_{2,2}, \ldots, x_{n,n}, \ldots$ is a rational *Cauchy* sequence, thus by Lemma 3.5, it converges to the real number

$$x := [x_{1,1}, x_{2,2}, \ldots, x_{n,n}, \ldots] \in \mathbb{R}.$$

By definition of convergence, there exists M' such that

$$|x_{n,n} - x| < \epsilon/3, \qquad \forall n \geq M'. \tag{3.10}$$

It follows that

$$\begin{aligned} \forall n \geq \max\{M, M'\}, \quad |x_n - y| &\leq |x_n - x_{n,n}| + |x_{n,n} - x| \quad \text{(triangle inequality)} \\ &< \epsilon/3 + \epsilon/3 \qquad\qquad \text{(Eq. (3.9), (3.10))} \\ &< \epsilon. \end{aligned}$$

Thus the real Cauchy sequence $x_1, x_2, \ldots$ is convergent, with limit $x \in \mathbb{R}$. □

Remark (not examinable): Both the Cauchy property and the convergence property make sense for sequences in a general metric space (a set X equipped with a distance function $d : X \times X \to \mathbb{R}$). Generally, a Cauchy sequence in X may not converge to anything in X (e.g., $X = \mathbb{Q}$). Nevertheless, there is a canonical way to embed X inside a larger metric space $\overline{X}$, called the *completion* of X, generalizing our construction of $\mathbb{R}$ from $\mathbb{Q}$. Cauchy sequences in the completion $\overline{X}$ will converge (to some limits in $\overline{X}$, so we say that $\overline{X}$ is *Cauchy complete*. The proof of Cauchy completeness $\overline{X}$ is basically the same as that for $\mathbb{R}$ in Proposition 3.7.

- Exercise: Recall the Zeno paradox. Show that the *geometric series* $T_1, T_2, \ldots$, where

$$T_n = \sum_{1 \leq k \leq n} \frac{1}{2^k},$$

is a Cauchy sequence.

Thus it is convergent by Proposition 3.7. What is its limit?

- There is another notion of "completeness", which is defined in terms of the concept of "least upper bound".

  Let A be a subset of $\mathbb{R}$. We say that $M \in \mathbb{R}$ is

  - an *upper bound* for A, if

    $$a \leq M, \qquad \forall a \in A; \tag{3.11}$$

  - a *least upper bound* for A, if it is an upper bound for A, and $M \leq M'$ holds for any other upper bound M' for A.

  Similarly, a real number M is a (greatest) *lower bound* for A iff $-M$ is a (least) upper bound for the set

  $$-A := \{-x \,:\, x \in A\}. \tag{3.12}$$

  Note that if A has a least upper bound, then it is unique (why?). Similarly, if A has a greatest lower bound, it is unique.

- The following set of rational real numbers,

  $$A = \{x \in \mathbb{R} \,:\, x \text{ rational and } x^2 < 2\},$$

  has rational upper bounds such as $2, \frac{3}{2}, \frac{29}{20}$. However, no *rational* number can be a *least* upper bound for A (try proving this as an exercise). We can, however, find a *real* least upper bound for A, due to the following fundamental property:

  Theorem 3.8 (Order completeness of $\mathbb{R}$)**.** *Any non-empty subset of* $\mathbb{R}$ *which has an upper bound, actually has a* least *upper bound.*

  (Similarly, if it has a lower bound, it has a greatest *lower bound.)*

  Proof. As A is non-empty, we may pick some $a \in A$. General observation: For any upper bound u and any non-upper-bound ℓ for A, we have $\ell < a \leq u$, thus

  $$\ell < u. \tag{3.13}$$

  By assumption we have some initial upper bound u_1 for A. Also, $\ell_1 := a - 1 < a$ is a *non*-upper-bound for A.

– By the general observation (3.13), $\ell_1 < u_1$. It is easily checked that the midpoint $\frac{\ell_1+u_1}{2}$ satisfies

$$\ell_1 < \frac{\ell_1 + u_1}{2} < u_1.$$

– Define a new upper bound $u_2 \leq u_1$ and a new non-upper-bound $\ell_2 \geq \ell_1$, by

$$\begin{cases} \ell_2 = \ell_1, \quad u_2 = \frac{\ell_1+u_1}{2}, & \text{if } \frac{\ell_1+u_1}{2} \text{ is an upper bound for } A, \\ \ell_2 = \frac{\ell_1+u_1}{2}, u_2 = u_1, & \text{if } \frac{\ell_1+u_1}{2} \text{ is not an upper bound for } A. \end{cases}$$

In either case,

$$u_2 - \ell_2 = \frac{1}{2}(u_1 - \ell_1)$$

– Similarly, we use the new midpoint $\frac{\ell_2+u_2}{2}$ either as a new upper bound $u_3 \leq u_2$, or as a new non-upper-bound $\ell_3 \geq \ell_2$. In either case,

$$u_3 - \ell_3 = \frac{1}{2}(u_2 - \ell_2) = \frac{1}{2^2}(u_1 - \ell_1).$$

– Recursively, we obtain a monotone decreasing sequence of upper bounds u_n, and a monotone increasing sequence of non-upper-bounds ℓ_n, such that

$$u_n - \ell_n = \frac{1}{2^{n-1}}(u_1 - \ell_1), \qquad \forall n \geq 1. \tag{3.14}$$

Note that because of (3.13), we have

$$\ell_1 \leq \ell_2 \leq \cdots \leq \ell_n \leq \cdots\cdots \leq u_n \leq \cdots \leq u_2 \leq u_1. \tag{3.15}$$

The next step is to show that the sequences $u_1, u_2, \ldots$ and $\ell_1, \ell_2, \ldots$ are Cauchy:

– Given a real $\epsilon > 0$, the Archimedean property (Proposition 3.1) says that there exists a positive integer N such that $N\epsilon > 4(u_1 - \ell_1)$. Also, an easy induction shows that $2^k \geq k$ for all positive integers k. Thus we have

$$4(u_1 - \ell_1) < 2^N \epsilon.$$

Putting this inequality into (3.14), we get

$$u_N - \ell_N < \frac{2^N \epsilon}{2^{N-1} \times 4} = \frac{\epsilon}{2}. \tag{3.16}$$

– The Cauchy property for $u_1, u_2, \ldots$ now follows:

$$\begin{aligned}
\forall m, n \geq N, \qquad |u_m - u_n| &\leq |u_m - \ell_N| + |\ell_N - u_n| && \text{(triangle inequality)} \\
&= (u_m - \ell_N) + (u_n - \ell_N) && \text{(Eq. (3.15))} \\
&\leq (u_N - \ell_N) + (u_N - \ell_N) && \text{(Eq. (3.15))} \\
&= 2(u_N - \ell_N) \\
&< \epsilon. && \text{(Eq. (3.16)).}
\end{aligned}$$

– Similarly, $\ell_1, \ell_2, \ldots$ is Cauchy.

By Theorem 3.8, $\lim_{n\to\infty} u_n$ and $\lim_{n\to\infty} \ell_n$ exist. Furthermore, formula (3.14) and Exercises 3.1, 3.6 show that

$$\lim_{n\to\infty} (u_n - \ell_n) = 0.$$

By Exercise 3.4,

$$0 = \lim_{n\to\infty} (u_n - l_n) = \lim_{n\to\infty} u_n - \lim_{n\to\infty} \ell_n.$$

Let us write the common limit as

$$s := \lim_{n\to\infty} u_n = \lim_{n\to\infty} \ell_n \in \mathbb{R}.$$

For the final step, we verify that s is the least upper bound for A.

– Proof of upper bound property by contradiction: Suppose $s < a$ for some $a \in A$, so $\epsilon = \frac{a-s}{2} > 0$. Then, since s is the limit of the non-increasing sequence $u_1, u_2, \ldots$, there exists some N such that

$$|u_N - s| < \epsilon = \frac{a-s}{2}.$$

We have $u_n - s \geq$ due to Exercise 3.3, so the above inequality becomes

$$u_N - a < \frac{s-a}{2} = -\epsilon < 0,$$

contradicting the upper bound property of u_N.

Thus $s \geq a$ holds for every $a \in A$, i.e., s is an upper bound for A.

– To show that s is the least upper bound, consider any $s' < s$. Since $s = \lim_{n\to\infty} \ell_n$ and $\epsilon = s - s' > 0$, there exists an N such that

$$|s - \ell_N| < \epsilon = s - s'.$$

We have $s - \ell_N \geq 0$ due to Exercise 3.3, so the above inequality becomes $s' < \ell_N$. By construction, ℓ_N is not an upper bound, so s' is also not an upper bound.

□

- Remarks:

  – The terminology "completeness of $\mathbb{R}$" often refers to the least upper bound property, Theorem 3.8. Some authors (e.g., Tao Section 5.5) first prove this (order) completeness directly, then deduce Cauchy completeness.

  Some textbooks may simply state the ordered field axioms for $\mathbb{R}$ without presenting a construction, then the order completeness property may be introduced without proof when needed (effectively, it is an extra axiom).

  – In the alternative "Dedekind cut" construction of $\mathbb{R}$ from $\mathbb{Q}$, the Archimedean and completeness properties are more easily proved, but the algebraic field properties are more tedious to verify (see Lou–Yang–Mei textbook, Chapter 2.3, if you are interested in this construction). After that, one introduces the convergence and limit concepts. The Cauchy sequence concept still needs to be introduced afterwards, then one proves the Cauchy completeness property.

  You may worry whether your friend from a different Analysis class, using the Dedekind cut construction of $\mathbb{R}$, is using the "same" $\mathbb{R}$ as we are. It turns out that all complete ordered fields are isomorphic (as ordered fields). So the *mathematics* of $\mathbb{R}$ (true statements and applicable tools) is independent of which construction is chosen. In fact, someone in the future might come up with yet another kind of construction of "$\mathbb{R}$" satisfying the complete ordered field property. Whether or not there is a "true" version of $\mathbb{R}$, is a matter of philosophy.

– Arguably, the axioms that any reasonable construction of $\mathbb{R}$ must satisfy, come from some concept of a *geometric* "real line". The aim is to obtain a number system "$\mathbb{R}$" which can provide *coordinates* for labelling points on the geometric line. It is helpful, especially if you study differential geometry or modern physics in the future, to keep in mind the conceptual difference between geometric objects and coordinate systems. In the old days, geometry was very successfully developed (e.g. Euclid, Newton) *synthetically*, without requiring any numerical coordinates whatsoever!