4 Lectures 7–8

Remark on notation: when we write "$\forall n \geq 1$", it should be clear from the context whether n refers to a positive integer labelling the terms of a sequence, or whether n refers to a general real/rational number.

- Theorem 3.8 only shows that certain subsets of $\mathbb{R}$ have least upper bounds or greatest lowest bounds. It is convenient to generalize these concepts to *arbitrary* subsets $A \subseteq \mathbb{R}$. We first introduce the set

$$\{-\infty\} \sqcup \mathbb{R} \sqcup \{+\infty\} \tag{4.1}$$

  of *extended real numbers*, with the $<$ relation extended to include

$$-\infty < a < +\infty, \qquad -\infty < +\infty, \qquad \forall a \in \mathbb{R}.$$

  We also write

$$-(+\infty) := -\infty, \qquad -(-\infty) := +\infty.$$

  For a subset $A \subseteq \mathbb{R}$, the *supremum* and *infimum* of A, denoted $\sup A$ and $\inf A$, respectively, are the extended real numbers defined as follows:

  - If $A \neq \emptyset$ and has an upper bound, then $\sup A \in \mathbb{R}$ is the least upper bound of A. (This exists by Theorem 3.8.)
  - If $A \neq \emptyset$ and has no upper bound, then[3] $\sup A := +\infty$.
  - If $A = \emptyset$, then[4] $\sup A := -\infty$.
  - The infimum of A is

$$\inf A = -\sup(-A).$$

  Note that

$$\begin{aligned} \inf A \leq \sup A, &\qquad A \neq \emptyset, \\ -\infty = \sup A < \inf A = +\infty, &\qquad A = \emptyset. \end{aligned}$$

[3] Formally, we allow $+\infty$ to be a "generalized upper bound", since $+\infty \geq a$ holds for all $a \in A$.

[4] For $A = \emptyset$, the upper bound condition (3.11) is an empty condition that is satisfied by *every* real number M, and also formally satisfied by $-\infty$ and $+\infty$. So we take the "smallest upper bound" $-\infty$ as the supremum.

- We will often refer to an *upper/lower bound of a sequence* of real numbers. This means an upper/lower bound for the (non-empty) *range* of the sequence. Likewise, the supremum/infimum of a sequence refers to the supremum/infimum of its range. Any sequence has a supremum/infimum (regarded as an extended real number). But a sequence may not have a least upper bound/greated lower bound (as a real number).

  A general sequence does not converge, so we should not expect its supremum/infimum to have any direct relationship with its limit behaviour. Specifically, the problem is that the ordering information of the terms is lost when we pass from the sequence to the range of the sequence.

  For example, the sequence $1, 0, 0, \ldots, 0, \ldots$ has supremum 1, and limit 0. In fact, all its subsequences converge to 0, so the supremum 1 has nothing to do with any limiting behaviour of the sequence.

  The sequence $-1, 1, -1, 1, \ldots, (-1)^n, \ldots$ does not have a limit. Its supremum 1 arises as the limit of the even subsequence.

  The sequence $-1, 1, -1, \frac{1}{2}, \ldots, -1, \frac{1}{n/2}, \ldots$ does not converge. Its supremum 1 is not the limit of any subsequence. Its infimum -1 is the limit of its odd subsequence. Its even subsequence converges to 0, which is somewhere in between the infimum and the supremum.

  Later, we will learn how to combine the supremum/infimum concept with the limit concept in a more meaningful way, via subsequences.

- As a warm-up, we investigate a special kind of sequence whose supremum/infimum is directly related to its limit.

  A sequence $x_1, x_2, \ldots$ of real numbers is *monotone* if

$$\forall n \geq 1, \qquad \begin{cases} x_n \leq x_{n+1}, & \text{(monotone increasing)} \\ x_n \geq x_{n+1}. & \text{(monotone decreasing)} \end{cases}$$

  Some examples of monotone sequences were encountered in the proof of Theorem 3.8.

  Theorem 4.1 (Monotone convergence theorem (MCT))**.** *If a sequence $x_1, x_2, \ldots$ of real numbers is monotone increasing and has an upper bound, then it converges to its supremum.*

  If it is monotone decreasing and has a lower bound, then it converges to its infimum.

Proof. It suffices to prove the monotone increasing case. Note that $\{x_n : n \geq 1\}$ is non-empty, and has an upper bound by assumption. By completeness, Theorem 3.8, the least upper bound of $\{x_n : n \geq 1\}$ exists; call it $x \in \mathbb{R}$.

As x is an upper bound, we have

$$0 \leq x - x_n = |x - x_n|, \qquad \forall n \geq 1.$$

As x is the *least* upper bound, given any $\epsilon > 0$, the number $x - \epsilon$ is not an upper bound, i.e., there exists some term x_N of the sequence, such that

$$x - \epsilon < x_N.$$

By the monotone increasing assumption,

$$x_N \leq x_n, \qquad \forall n \geq N.$$

Combining the three statements in the above paragraph,

$$|x - x_n| = x - x_n \leq x - x_N < \epsilon, \qquad \forall n \geq N.$$

This shows that the sequence converges to x, its least upper bound (i.e., supremum). □

- Here are some examples to illustrate the MCT.

  Example 4.1. The sequences in Example 3.1 are monotone decreasing, and bounded below by 0. In fact, 0 is easily checked to be the greatest lower bound. Thus, by MCT, these sequences converge to 0 (we already verified this directly).

  Exercise 4.1. Suppose $0 \leq \lambda < 1$. Show that

  $$\lim_{n\to\infty} \lambda^n = 0.$$

  Example 4.2. You may have seen the "continued fraction" formula

  $$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2+\ldots}}} \tag{4.2}$$

A "proof" is:

$$x := 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}} \quad \text{satisfies} \quad x = 2 + \frac{1}{x},$$

then solving the quadratic equation for x gives $x = \sqrt{2} + 1$. Actually, we need to first argue that the continued fraction expression is a well-defined real number. This is the $x_1 = 2$ case of the discussion below.

Start with any positive real number $x_1 > 0$, and recursively define

$$x_{n+1} = 2 + \frac{1}{x_n}, \qquad n \geq 1.$$

Note that by induction,

$$x_n > 2, \qquad \forall n \geq 2,$$

(thus $\frac{1}{x_n}$ exist), and

$$2 < x_n < 2 + \frac{1}{2}, \qquad \forall n \geq 3.$$

So the sequence $x_1, x_2, \ldots, x_n, \ldots$ has an upper bound and a lower bound. To make use of the MCT, we need some monotonicity properties. We will argue that the even subsequence and the odd subsequence are monotone.

Observe that

$$\begin{aligned} \forall n \geq 1, \qquad x_{n+4} - x_{n+2} &= 2 + \frac{1}{x_{n+3}} - (2 + \frac{1}{x_{n+1}}) \\ &= \frac{x_{n+1} - x_{n+3}}{x_{n+1} x_{n+3}} \\ &= \frac{(2 + \frac{1}{x_n}) - (2 + \frac{1}{x_{n+2}})}{x_{n+1} x_{n+3}} \\ &= \frac{x_{n+2} - x_n}{x_n x_{n+1} x_{n+2} x_{n+3}}. \end{aligned} \tag{4.3}$$

Since the denominator of (4.3) is positive, we know that $x_{n+4} - x_{n-2}$ has the same sign as $x_{n-2} - x_n$. Thus, the odd subsequence $y_1, y_2, \ldots$ and the even subsequence $z_1, z_2, \ldots$,

$$y_k := x_{2k-1}, \qquad z_k := x_{2k}, \qquad k \geq 1,$$

are monotone. By MCT, the following limits exist

$$\ell := \lim_{k\to\infty} y_k, \qquad L := \lim_{k\to\infty} z_k.$$

These limits are positive, because of the lower bound of 2 for the sequence.

Now, $z_k = x_{2k} = 2 + \frac{1}{x_{2k-1}} = 2 + \frac{1}{y_k}$. Therefore, by algebraic properties of limits,

$$L = \lim_{k\to\infty} z_k = 2 + \frac{1}{\lim_{k\to\infty} y_k} = 2 + \frac{1}{\ell}.$$

Similarly, $y_{k+1} = x_{2k+1} = 2 + \frac{1}{x_{2k}} = 2 + \frac{1}{z_k}$, so

$$\ell = \lim_{k\to\infty} y_k = \lim_{k\to\infty} y_{k+1} = 2 + \frac{1}{L}.$$

The simultaneous equations in L, ℓ are solved by

$$L = \sqrt{2} + 1 = \ell.$$

(The other solution is inadmissible because the lower bound implies $L \geq 2$.)

Finally, use Exercise 3.5 to deduce that

$$\lim_{n\to\infty} x_n = \sqrt{2} + 1.$$

Remark. Starting with $x_1 = 2$, you may verify that the odd subsequence is monotone increasing, while the even subsequence is monotone decreasing. So the original sequence is not monotone. Subtracting 1, we obtain a concrete example of a rational Cauchy sequence converging to $\sqrt{2}$. Previously, we only argues that such a rational Cauchy sequence exists.

- Let n be a natural number. Recall the recursive definition of x^n, where $x \in \mathbb{N}$. The same formula is used for real numbers $x \in \mathbb{R}$. We wish to invert this procedure.

  Lemma 4.2 (Existence of n-th roots)**.** *Let x be a non-negative real number. For each positive integer $n \geq 1$, there exists a unique non-negative real number y satisfying*

  $$y^n = x,$$

namely,

$$x^{1/n} \equiv \sqrt[n]{x} := \sup\{a \in \mathbb{R} : 0 \leq a \text{ and } a^n \leq x\} \in \mathbb{R}. \tag{4.4}$$

If x is positive, then $x^{1/n}$ is positive.

Proof. Write A for the set (4.4). It is non-empty since it contains 0. We check that A has an upper bound.

- $x \leq 1$ case. Suppose there exists some $a \in A$ with $1 < a$. Then $x \leq 1 = 1^n < a^n$, contradicting $a \in A$.
- $1 < x$ case. We argue that x is an upper bound for A. Suppose otherwise, that there exists some $a \in A$ with $x < a$. So $1 < x < a$. Thus $a < a^2$, and $a^2 < a^3$, etc., so $x < a < a^n$. This contradicts $a \in A$.

By the completeness property (Theorem 3.8), A has a lowest upper bound, i.e., the supremem of A exists in $\mathbb{R}$.

The statement that $x^{1/n}$ is the unique non-negative solution to $y^n = x$ is left as an exercise.

If $x^{1/n} = 0$, then $x = 0^n = 0$. So if $x > 0$, we must have $x^{1/n} > 0$ as well. □

In the $n = 2$ case, we usually just write $\sqrt{x} \equiv x^{1/2}$. Check that our previous definition of $\sqrt{2}$ in terms of a certain rational Cauchy sequence (satisfying Eq. (2.1)) is consistent with the definition (4.4).

Exercise 4.2. Let $x > 0$ be a positive real number. Show that

$$\lim_{n\to\infty} x^{1/n} = 1.$$

- (Exponentiation I.) Let x be a *non-zero* real number. We can exponentiate it by an *integer* exponent $n \in \mathbb{Z}$, by setting

$$x^{-n} := \frac{1}{x^n}, \qquad n \in \mathbb{N}, x \in \mathbb{R} \setminus \{0\}. \tag{4.5}$$

Let $x \geq 0$ be a *non-negative* real number, and let n be a positive integer. In Lemma 4.2, we explained that it has a unique non-negative n-th root $x^{1/n}$.

Let $x > 0$ be a *positive* real number. We can combine the n-th root construction with the integer exponent definition (4.5), as follows. Let $q \in \mathbb{Q}$ be *rational*, and write $q = \frac{a}{b}$ where a is an integer and b is a *positive* integer. Then define

$$x^q := (x^{\frac{1}{b}})^a, \qquad x > 0,\, q = \frac{a}{b} \in \mathbb{Q}. \quad (b \geq 1). \tag{4.6}$$

As an exercise, check that formula (4.6) is independent of the choice of numerator a and positive denominator b when representing q as $\frac{a}{b}$.

For positive real numbers $x, y > 0$ and $p, q \in \mathbb{Q}$, we list a few basic properties of exponentiation (see Tao's book Chapter 5.6):

- We have
$$x^q > 0, \qquad 1^q = 1. \tag{4.7}$$
- The familiar algebraic rules of exponentiation hold:
$$x^{p+q} = x^p x^q, \qquad (x^p)^q = x^{pq}, \qquad (xy)^q = x^q y^q. \tag{4.8}$$
- When $x > 1$ and $q > 0$,
$$x^q > 1. \tag{4.9}$$
- When $x > 1$,
$$p < q \;\Leftrightarrow\; x^p < x^q. \tag{4.10}$$
When $0 < x < 1$, we have $x^{-1} > 1$, then using $(x^{-1})^q = x^{-q}$ and (4.10), we get
$$p < q \;\Leftrightarrow\; x^q < x^p. \tag{4.11}$$

You may try to prove the above statements from the definitions.

- (Exponentiation II.) To define x^α for a *real* number α (and $x > 0$) we have to use a limiting procedure.

Lemma 4.3. *Let $x > 0$ be a positive real number, and $\alpha \in \mathbb{R}$. For any rational Cauchy sequence $q_1, q_2, \ldots$ representing α, the following limit exists,*

$$\lim_{n\to\infty} x^{q_n}. \tag{4.12}$$

Proof. Let M be a rational lower bound for the rational Cauchy sequence representing α. We prove the lemma in the $0 < x < 1$ case; the $x > 1$ case is similar, and is handled in Lemma 6.7.1 of Tao's book.

Let $\epsilon > 0$, thus $x^{-M}\epsilon > 0$ be given. As $0 < x < 1$, we have $x^{-1} > 1$, thus $x^{-1/k} > 1$, and

$$|x^{-1/k} - 1| = x^{-1/k} - 1, \qquad \forall k \geq 1.$$

In Exercise 4.2, we found that $\lim_{k\to\infty} x^{1/k} = 1$, then by quotient rule for limits (why does it apply?),

$$\lim_{k\to\infty} x^{-1/k} = 1$$

as well. So there exists some $K \geq 1$ such that

$$x^{-1/K} - 1 = |x^{-1/K} - 1| < \epsilon x^{-M}. \tag{4.13}$$

Also, the Cauchy property of $q_1, q_2, \ldots$ implies that there exists $N \geq 1$ such that

$$|q_m - q_n| < \frac{1}{K}, \qquad \forall m, n \geq N. \tag{4.14}$$

Our aim is to show that

$$|x^{q_m} - x^{q_n}| < \epsilon, \qquad \forall m, n \geq N, \tag{4.15}$$

i.e., $x^{q_1}, x^{q_2}, \ldots$ is Cauchy, thus it is convergent by the Cauchy completeness theorem.

It suffices to assume that $q_m \geq q_n$ in (4.15); equivalently, by (4.11),

$$x^{q_n} \geq x^{q_m}.$$

(Otherwise, if $q_m \leq q_n$, exchange the roles of m, n in the calculations below).

We first show that assume (4.15) holds under the extra assumption

$$q_m \geq q_n \overset{(4.11)}{\Leftrightarrow} x^{q_n} \geq x^{q_m}. \tag{4.16}$$

In this case, the inequality (4.14) becomes

$$\forall m, n \geq N \text{ with } q_m \geq q_n, \qquad q_m - q_n < \frac{1}{K} \overset{(4.11)}{\Leftrightarrow} x^{q_n - q_m} < x^{-1/K}. \tag{4.17}$$

Thus,

$$\begin{aligned}
\forall m, n \geq N \text{ with } q_m \geq q_n, \qquad & |x^{q_m} - x^{q_n}| \\
& = x^{q_n} - x^{q_m} && \text{(by (4.16))} \\
& = x^{q_m}(x^{q_n - q_m} - 1) && \text{(by (4.8))} \\
& < x^M(x^{-1/K} - 1) && (M \leq q_m, (4.11), \text{ and } (4.17)) \\
& \leq x^M x^{-M} \epsilon && \text{(by (4.13))} \\
& = \epsilon.
\end{aligned}$$

The $q_m \leq q_n$ case of (4.15) is proved similarly: exchange the roles of m, n in the above calculations. Thus the Cauchy property claim, Eq. (4.15) is verified. □

Lemma 4.4. *The limit* (4.12) *is independent of the choice of rational Cauchy sequence* $q_1, q_2, \ldots$ *representing* $\alpha \in \mathbb{R}$.

Proof. Suppose $q_1', q_2', \ldots$ is another rational Cauchy sequence representing $\alpha \in \mathbb{R}$. Let $\epsilon > 0$ be given. In the proof of Lemma 4.3, we mentioned that

$$\lim_{k\to\infty} x^{1/k} = 1 = \lim_{k\to\infty} x^{-1/k},$$

thus there exists some $K \geq 1$ such that

$$|x^{1/K} - 1| < \epsilon \quad \text{and} \quad |x^{-1/K} - 1| < \epsilon.$$

Also, the sequence $r_1, r_2, \ldots$ with $r_n := q_n - q_n'$ converges to 0. So there exists $N \equiv N(1/K)$ such that

$$-1/K < r_n < 1/K, \qquad \forall n \geq N.$$

– If $x > 1$, we deduce (using (4.10)) that

$$\forall n \geq N \text{ with } r_n \geq 0, \qquad |x^{r_n} - 1| = x^{r_n} - 1 < x^{1/K} - 1 = |x^{1/K} - 1| < \epsilon,$$

and

$$\forall n \geq N \text{ with } r_n < 0, \qquad |x^{r_n} - 1| = 1 - x^{r_n} < 1 - x^{-1/K} = |x^{-1/K} - 1| < \epsilon.$$

Thus $\lim_{n\to\infty} x^{r_n} = 1$.

– If $0 < x < 1$, we deduce (using (4.11)) that

$\forall n \geq N$ with $r_n \geq 0$, $\qquad |x^{-r_n} - 1| = x^{-r_n} - 1 < x^{-1/K} - 1 = |x^{-1/K} - 1| < \epsilon$,

and

$\forall n \geq N$ with $r_n < 0$, $\qquad |x^{-r_n} - 1| = 1 - x^{-r_n} < 1 - x^{1/K} = |x^{1/K} - 1| < \epsilon$.

So, we again have $\lim_{n\to\infty} x^{r_n} = 1/\lim_{n\to\infty} x^{-r_n} = 1/1 = 1$.

It follows that

$$\lim_{n\to\infty} x^{q_n} = \lim_{n\to\infty} x^{r_n + q'_n} \overset{(4.8)}{=} \lim_{n\to\infty} x^{r_n} x^{q'_n} \overset{\text{Exercise 3.4}}{=} \left(\lim_{n\to\infty} x^{r_n}\right)\left(\lim_{n\to\infty} x^{q'_n}\right) = \lim_{n\to\infty} x^{q'_n}.$$

□

Because of Lemma 4.3 and Lemma 4.4, the following exponentiation formula is well-defined,

$$x^q := \lim_{n\to\infty} x^{q_n}, \qquad q = [q_1, q_2, \ldots] \in \mathbb{R},\ x > 0. \tag{4.18}$$

Exercise 4.3. Verify that (4.7)–(4.11) still hold for real exponents $p, q \in \mathbb{R}$.

Exercise 4.4. Let $x, \alpha \in \mathbb{R}$ with $x \geq 1$. Show that

$$x^\alpha = \sup\{x^q \,:\, q \in \mathbb{Q}, q \leq \alpha\}.$$

Remark: Some authors use the expression on the right side as the definition of x^α (when $x \geq 1$).

Exercise 4.5. Let $x, y > 0$ be positive real numbers. Suppose $x \neq 1$. Show that there exists a unique real number α such that $x^\alpha = y$.

Remark: The number α is usually denoted $\alpha = \log_x y$.

- The MCT can also be used to define more interesting irrational numbers which do not arise from taking n-th roots. (Such real numbers are not solutions to algebraic equations.)

  For any two non-negative real numbers a, b,

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

  The proof of this case is quite straightforward (do it yourself). This inequality is the famous arithmetic mean–geometric mean (AM–GM) inequality, and it generalizes:

Lemma 4.5 (AM–GM inequality)**.** *Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Then*

$$(a_1 a_2 \ldots a_n)^{1/n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n},$$

with equality iff $a_1 = a_2 = \ldots = a_n$.

As there are cleaner proofs of Lemma 4.5 using more calculus techniques, we omit the proof here, and just refer to [Lou–Yang–Mei textbook 2.4.2] for an "elementary[5]" induction proof.

The AM–GM inequality, Lemma 4.5, can be rewritten as

$$a_1 a_2 \ldots a_n \leq \left(\frac{a_1 + a_2 + \ldots + a_n}{n} \right)^n,$$

with strict inequality unless all the terms $a_1, \ldots, a_n$ are equal. Then

$$(1+\frac{1}{n})^n = \overbrace{(1+\frac{1}{n}) \cdot (1+\frac{1}{n})}^{n \text{ times}} \cdot (1) < \left(\frac{n(1+\frac{1}{n})+1}{n+1} \right)^{n+1} = (1+\frac{1}{n+1})^{n+1}.$$

Thus the sequence $x_1, x_2, \ldots$ with

$$x_n = (1+\frac{1}{n})^n$$

is monotone increasing.

A similar use of the AM–GM inequality, and the well-known formula (Exercise),

$$\frac{1}{x^n} = (\frac{1}{x})^n, \qquad x \neq 0,\ n \in \mathbb{N}, \tag{4.19}$$

gives

$$(1+\frac{1}{n})^{n+1} = 1/(\frac{n}{n+1})^{n+1} = 1/\big(\underbrace{\frac{n}{n+1} \cdots \frac{n}{n+1}}_{n+1 \text{ times}} \cdot 1 \big) > 1/(\frac{n+1}{n+2})^{n+2}$$
$$= (1+\frac{1}{n+1})^{n+2}.$$

[5] "Elementary" means "requiring less theory". It does not mean "easy"!

So the sequence $y_1, y_2, \ldots$ with

$$y_n = (1 + \frac{1}{n})^{n+1}$$

is monotone decreasing. The initial term $y_1 = 4$ is an upper bound.

Since

$$x_n \leq x_n \cdot (1 + \frac{1}{n}) = y_n \leq 4, \qquad \forall n \geq 1,$$

the monotone increasing sequence $x_1, x_2, \ldots$ has an upper bound. By MCT, it has a limit. This limit is one possible definition of *Euler's number*,

$$e := \lim_{n\to\infty} (1 + \frac{1}{n})^n = \sup \Big\{ (1 + \frac{1}{n})^n \,:\, n \geq 1 \Big\}. \tag{4.20}$$

We will see a few other equivalent definitions of e later on.

Exercise 4.6. Continue the above argument to deduce that

$$e = \lim_{n\to\infty} (1 + \frac{1}{n})^{n+1} = \inf \Big\{ (1 + \frac{1}{n})^{n+1} \,:\, n \geq 1 \Big\}.$$

- The MCT requires the very strong condition of monotonicity. Actually, monotone sequences are very common in the sense of *sub*sequences.

  Lemma 4.6. *Every sequence of real numbers has a monotone subsequence.*

  Proof. Let $x_1, x_2, \ldots$ be a sequence of real numbers. We say that a term x_n is a *peak*, if

  $$x_m \leq x_n, \qquad \forall m > n.$$

  Suppose there are infinitely many peaks. Arranging the peaks in the order in which they occur, we get a subsequence

  $$x_{n_1} \geq x_{n_2} \geq \cdots \geq x_{n_k} \geq \cdots, \qquad k \geq 1.$$

  which is monotone decreasing by definition of "peak".

  Otherwise, suppose there are finitely many peaks, and let x_N be the last peak. So

  $$x_{n_1} := x_{N+1}$$

is not a peak. This means that there exists some $n_2 > n_1$ such that

$$x_{n_1} < x_{n_2}.$$

Again, x_{n_2} is not a peak, so there exists some $n_3 > n_2$ such that

$$x_{n_2} < x_{n_3}.$$

Recursively, we obtain a subsequence

$$x_{n_1} < x_{n_2} < \cdots < x_{n_k} < \cdots, \qquad k \geq 1,$$

which is monotone increasing. □

Specializing Lemma 4.6 to *bounded* sequences, we easily obtain, from the MCT, a useful result.

Theorem 4.7 (Bolzano–Weierstrass Theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. By Lemma 4.6, there is some monotone subsequence. By assumption, the original sequence, thus also the subsequence, has both an upper bound and a lower bound. Whether the monotone subsequence is increasing or decreasing, we can apply the MCT to deduce that it converges. □

Remark: Theorem 4.7 generalizes to $\mathbb{R}^d$-valued sequences, as you will learn in future courses.

- The Bolzano–Weierstrass Theorem does not provide an explicit convergent subsequence, and also does not tell us what the limit of such a convergent subsequence is. (We will improve the situation later, see future Remark 1.)

  Nevertheless, the Bolzano–Weierstrass Theorem guarantees that a bounded sequence of real numbers has a *non-empty* set of *sub*sequential limits. Here, a real number is a *subsequential limit* if it is the limit of some (convergent) subsequence.

  Therefore, by the completeness Theorem 3.8, the following two real numbers are well-defined for any bounded sequence:

$$\begin{aligned} x^+ &:= \sup\{\text{subsequential limits}\} \in \mathbb{R}, \\ x^- &:= \inf\{\text{subsequential limits}\} \in \mathbb{R}. \end{aligned} \tag{4.21}$$

- Eq. (4.21) defines the "sup-sublim" and "inf-sublim" of a bounded sequence. Below, we introduce the "lim-subsup" and "lim-subinf" concepts. Then we will show they coincide with "sup-sublim" and "inf-sublim" respectively (Proposition 5.1).

  Given a *bounded* sequence $x_1, x_2, \ldots$ of real numbers, let us define

$$\begin{aligned} x_n^+ &:= \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} \equiv \sup_{k\geq n} x_k, \\ x_n^- &:= \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\} \equiv \inf_{k\geq n} x_k, \quad n \geq 1. \end{aligned} \tag{4.22}$$

  These numbers are "sub-supremums" and "sub-infimums". (Note: The sets $\{x_n, x_{n+1}, x_{n+2}, \ldots\}$ have upper/lower bounds because $\{x_1, x_2, \ldots\}$ is bounded.) Then we obtain the *monotone* sequences

$$\begin{aligned} &x_1^+, x_2^+, \ldots, x_n^+, \ldots, \qquad \text{(decreasing)} \\ &x_1^-, x_2^-, \ldots, x_n^-, \ldots. \qquad \text{(increasing)} \end{aligned}$$

  which satisfy

$$x_n^- \leq x_n^+, \qquad \forall n \geq 1. \tag{4.23}$$

  Note, however, that $x_1^\pm, x_2^\pm, \ldots$ is *not* necessarily a subsequence of $x_1, x_2, \ldots$.

  Because

$$x_1^- \leq x_n^- \leq x_n^+ \leq x_1^+, \qquad \forall n \geq 1,$$

  the monotone decreasing sequence $x_1^+, x_2^+, \ldots$ has a lower bound, while the monotone increasing sequence $x_1^-, x_2^-, \ldots$ has an upper bound. So we may apply MCT to obtain the following two limits,

$$\begin{aligned} \overline{\lim_{n\to\infty}}\, x_n &:= \lim_{n\to\infty} x_n^+ \equiv \lim_{n\to\infty} \sup_{k\geq n} x_k, \\ \underline{\lim_{n\to\infty}}\, x_n &:= \lim_{n\to\infty} x_n^- \equiv \lim_{n\to\infty} \inf_{k\geq n} x_k. \end{aligned} \tag{4.24}$$

  The two limits in (4.24) are, respectively, called the *limit superior* and *limit inferior*[6] of the sequence $x_1, x_2, \ldots$. They are pronounced "*lim-sup*" and "*lim-inf*".

- Somewhat more generally, the x_n^+, thus the lim-sup, is defined when the sequence has an upper bound, while the x_n^-, thus the lim-inf, is defined when the sequence has a lower bound.

[6] Another common notation is $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.

Exercise 4.7. Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be two sequences of real numbers, such that

$$x_n \leq y_n, \qquad \forall n \geq 1.$$

– If the sequences converge, show that

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

In particular, the limit of a convergent (thus bounded) sequence lies between any lower bound and any upper bound.

– If $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ have lower bounds, show that

$$\underline{\lim}_{n\to\infty} x_n \leq \underline{\lim}_{n\to\infty} y_n,$$

If $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ have upper bounds, show that

$$\overline{\lim}_{n\to\infty} x_n \leq \overline{\lim}_{n\to\infty} y_n.$$

Proposition 4.8. *Let $x_1, x_2, \ldots$ be a bounded sequence of real numbers. Then*

$$\underline{\lim}_{n\to\infty} x_n \leq \overline{\lim}_{n\to\infty} x_n. \tag{4.25}$$

Equality in (4.25) *holds iff the limit* $\lim_{n\to\infty} x_n$ *exists, in which case, they are equal to the limit.*

Proof. Because of the term-wise inequalities $x_n^- \leq x_n^+$ (Eq. (4.23)), the respective limits also satisfy (Exericse 4.7)

$$\underline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} x_n^- \leq \lim_{n\to\infty} x_n^+ = \overline{\lim}_{n\to\infty} x_n.$$

($\Rightarrow$) Now suppose $\underline{\lim}_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n$ exist as well. Because we actually have

$$x_n^- \leq x_n \leq x_n^+, \qquad n \geq 1,$$

by the squeezing principle, $\lim_{n\to\infty} x_n$ exists and is equal to the common limit of x_n^- and x_n^+.

($\Leftarrow$) Conversely, suppose $L = \lim_{n\to\infty} x_n$ exists. Given $\epsilon > 0$, we can find some N such that

$$L - \epsilon < x_k < L + \epsilon, \qquad \forall k \geq n \geq N.$$

Taking infinum/supremum over $k \geq n$,

$$L - \epsilon \leq x_n^- \leq x_n^+ \overset{(4.23)}{\leq} L + \epsilon, \qquad \forall n \geq N.$$

Taking the difference of the $n \to \infty$ limits (and using Exercise 3.3), we have

$$\overline{\lim_{n\to\infty}}\, x_n - \underline{\lim_{n\to\infty}}\, x_n \leq (L+\epsilon) - (L-\epsilon) = 2\epsilon.$$

This inequality holds for any given $\epsilon > 0$. Thus we must have

$$\overline{\lim_{n\to\infty}}\, x_n = \underline{\lim_{n\to\infty}}\, x_n.$$

□

- Sometimes, one allows for the concept of "convergence to $+\infty$". This means that given any $\mu \in \mathbb{R}$, there exists $N \geq 1$ such that $x_n > \mu$ for all $n \geq N$. Likewise for "convergence to $-\infty$". Some authors say "diverge to $+\infty$" or "diverge to $-\infty$" to emphasize that there is actually no convergence in the usual real number sense. One can then consider sequences whose terms lie in the set of *extended* real numbers. The supremum/infimum of such a sequence is an extended real number, defined in the obvious way.

  We will seldom need to use such a generalization. We just mention that an advantage of the generalization is that the lim-inf and lim-sup of any (not necessarily bounded) sequence always exist as extended real numbers. You may also check that the results of Exercise 4.7 and Prop. 4.8 generalize to (generally unbounded) sequences of extended real numbers.