5 Lectures 9–10

- Consider the sequence

$$-(1+1), (1+\frac{1}{2}), -(1+\frac{1}{3}), (1+\frac{1}{4}), \ldots, (-1)^n(1+\frac{1}{n}), \ldots. \tag{5.1}$$

The sequence of "subsups" is

$$1+\frac{1}{2}, 1+\frac{1}{2}, 1+\frac{1}{4}, 1+\frac{1}{4}, 1+\frac{1}{6}, 1+\frac{1}{6}, \ldots,$$

so the lim-sup is 1. The sequence of "subinfs" is

$$-(1+1), -(1+\frac{1}{3}), -(1+\frac{1}{3}), -(1+\frac{1}{5}), -(1+\frac{1}{5}), \ldots$$

so the lim-inf is -1, which is different from the lim-sup. By Prop. 4.8, the sequence (5.1) does not converge.

The odd subsequence converges to -1 and the even subsequence converges to $+1$. (This is another way to show that the sequence does not converge.) In fact, we can argue that no other number can be obtained as a subsequential limit. So, in the notation of (4.21), we have

$$x^+ = 1, \qquad x^- = -1.$$

These coincide with the lim-sup and lim-inf respectively.

- The previous example exhibits a special case of the following result.

Proposition 5.1. *Let $x_1, x_2, \ldots$ be a bounded sequence of real numbers. Then*

$$\overline{\lim_{n\to\infty}}\, x_n = \sup\{\text{subsequential limits}\} \equiv x^+,$$
$$\underline{\lim_{n\to\infty}}\, x_n = \inf\{\text{subsequential limits}\} \equiv x^-.$$

Proof. It will suffice to prove the first equation (the second equation is proved similarly). Equivalently, we need to show the two inequalities,

$$\overline{\lim_{n\to\infty}}\, x_n \geq x^+ \quad \text{and} \quad \overline{\lim_{n\to\infty}}\, x_n \leq x^+.$$

– Let $x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots$ be a convergent subsequence of $x_1, x_2, \ldots$. Certainly, we have the inequalities

$$x^+_{n_k} = \sup\{x_{n_k}, x_{n_k+1}, x_{n_k+2}, \ldots\} \geq x_{n_k}, \qquad \forall k \geq 1. \tag{5.2}$$

The left side is the k-th term of

$$x^+_{n_1}, x^+_{n_2}, \ldots, x^+_{n_k}, \ldots,$$

a subsequence of the convergent sequence $x^+_1, x^+_2, \ldots$. Thus

$$\lim_{n\to\infty} x^+_n = \lim_{k\to\infty} x^+_{n_k}.$$

By assumption, the $k \to \infty$ limit of the right side of (5.2) also exists. Since limits respect order (Exercise 4.7), we have

$$\overline{\lim_{n\to\infty}}\, x_n \equiv \lim_{n\to\infty} x^+_n = \lim_{k\to\infty} x^+_{n_k} \geq \lim_{k\to\infty} x_{n_k}.$$

This inequality is valid for every convergent subsequence $x_{n_1}, x_{n_2}, \ldots$ of $x_1, x_2, \ldots$. Consequently,

$$\overline{\lim_{n\to\infty}}\, x_n \geq \sup\{\text{subsequential limits}\} \equiv x^+.$$

– The reverse inequality is harder. Let us recursively construct a subsequence $x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots$ with the property that

$$x^+_{n_k+1} - \tfrac{1}{k} < x_{n_{k+1}} \leq x^+_{n_k+1}, \qquad k \geq 1. \tag{5.3}$$

First, set $x_{n_1} = x_1$. Next, fix $k \geq 1$ and suppose x_{n_k} has been defined. We define $x_{n_{k+1}}$ satisfying (5.3) as follows. By definition, $x^+_{n_k+1}$ is the least upper bound for the set

$$\{x_{n_k+m} \,:\, m \in \mathbb{N} \setminus \{0\}\}. \tag{5.4}$$

So $x^+_{n_k+1} - \frac{1}{k}$ is not an upper bound for (5.4). Therefore, the set

$$\left\{m \in \mathbb{N} \setminus \{0\} \,:\, x^+_{n_k+1} - \tfrac{1}{k} < x_{n_k+m} \leq x^+_{n_k+1}\right\} \tag{5.5}$$

is non-empty. Define $x_{n_{k+1}} = x_{n_k+m_k}$ where m_k is the smallest element of (5.5), so $x_{n_{k+1}}$ satisfies (5.3).

We now have the following three sequences,

$$\begin{array}{rcccccc} x^+_{n_1+1} - \frac{1}{1}, & & x^+_{n_2+1} - \frac{1}{2}, & \ldots, & x^+_{n_k+1} - \frac{1}{k}, & \ldots, \\ x_{n_2} & , & x_{n_3} & \ldots, & x_{n_{k+1}} & \ldots, \\ x^+_{n_1+1} & , & x^+_{n_2+1} & \ldots, & x^+_{n_k+1} & \ldots, \end{array}$$

whose k-th terms ($k \geq 1$) are related by the inequalities (5.3). The third sequence is a subsequence of the convergent sequence $x_1^+, x_2^+, \ldots$, thus by Exercise 3.5, it has the same limit,

$$\lim_{k\to\infty} x^+_{n_k+1} = \lim_{n\to\infty} x_n^+ \equiv \overline{\lim_{n\to\infty}}\, x_n.$$

The first sequence is obtained from the third sequence by adding the 0-limit sequence $-\frac{1}{1}, -\frac{1}{2}, \ldots, -\frac{1}{k}, \ldots$. So the first sequence also converges to $\overline{\lim}_{n\to\infty} x_n$. By the squeezing principle, the second sequence must converge to $\overline{\lim}_{n\to\infty} x_n$ as well, i.e.,

$$\lim_{k\to\infty} x_{n_k} = \overline{\lim_{n\to\infty}}\, x_n. \tag{5.6}$$

Now, the sequence $x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots$ is actually a *subsequence* of $x_1, x_2, \ldots$, because $n_{k+1} = n_k + m_k > n_k$ for all $k \geq 1$. So (5.6) says that the number $\overline{\lim}_{n\to\infty} x_n$ is a subsequential limit. Therefore,

$$\overline{\lim_{n\to\infty}}\, x_n \leq \sup\{\text{subsequential limits}\} \equiv x^+.$$

□

Remark 1. Notice that in the second part of the proof of Prop. 5.1, we constructed a special subsequence that converges to the lim-sup. Similarly, we may construct a subsequence which converges to the lim-inf. Therefore, Prop. 5.1 implies a slightly stronger version of the Bolzano–Weierstrass Theorem: a bounded sequence has a (not necessarily monotone) subsequence which converges to the lim-sup. (It also has one which which converges to the lim-inf).

- The lim-inf/sup operation is not strictly additive. For example, we only have an inequality

$$\inf_{k\geq n} a_k + \inf_{k\geq n} b_k \leq \inf_{k\geq n}(a_k + b_k),$$

so after taking limits, we can only deduce that lim-inf is *sub*additive,

$$\varliminf_{n\to\infty} a_n + \varliminf_{n\to\infty} b_n \le \varliminf_{n\to\infty} (a_n + b_n).$$

Another tricky issue is that $\sup(-A) = -\inf(A)$, so

$$\varlimsup_{n\to\infty} (-a_n) = -\varliminf_{n\to\infty} a_n.$$

Exercise 5.1. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of real numbers. Show that

$$\begin{aligned}\varliminf_{n\to\infty} a_n + \varliminf_{n\to\infty} b_n &\le \varliminf_{n\to\infty} (a_n + b_n) \\ &\le \varliminf_{n\to\infty} a_n + \varlimsup_{n\to\infty} b_n \le \varlimsup_{n\to\infty} (a_n + b_n) \le \varlimsup_{n\to\infty} a_n + \varlimsup_{n\to\infty} b_n,\end{aligned}$$

where each inequality is valid whenever the lim-inf/lim-sup on each side exist.

Similarly, if $a_n, b_n > 0$ for all $n \ge 1$, we have

$$\begin{aligned}\varliminf_{n\to\infty} a_n \varliminf_{n\to\infty} b_n &\le \varliminf_{n\to\infty} a_n b_n \\ &\le \varliminf_{n\to\infty} a_n \varlimsup_{n\to\infty} b_n \\ &\le \varlimsup_{n\to\infty} a_n b_n \le \varlimsup_{n\to\infty} a_n \varlimsup_{n\to\infty} b_n,\end{aligned}$$

as well as

$$\varlimsup_{n\to\infty} 1/a_n = 1/\varliminf_{n\to\infty} a_n.$$

- Here is an example where the existence of the limit is difficult to show directly. Instead, we show equality of lim-inf and lim-sup to establish the existence of limit.

  Example 5.1 (Adapted from Example 3.4.2 of Lou–Yang–Mei, and sketched in class). Let $x_1, x_2, \ldots$ be a sequence of positive real numbers satisfying

  $$x_{m+n} \le \frac{m x_m + n x_n}{m+n}, \qquad \forall m, n \ge 1. \tag{5.7}$$

  Check by induction that

  $$x_n \le 1, \qquad \forall n \ge 1,$$

thus the sequence is bounded; more generally,

$$x_{mn} \le x_n, \qquad \forall m, n \ge 1. \tag{5.8}$$

For $m \ge n \ge 1$, use Euclid's division lemma to write

$$m = q_m n + r_m, \qquad q_m, r_m \in \mathbb{N},\ 0 \le r_m \le n-1,\ q_m \ge 1.$$

Then

$$\begin{aligned} x_m = x_{q_m n + r_m} &\overset{(5.7)}{\le} \frac{q_m n x_{q_m n} + r_m x_{r_m}}{m} \\ &\overset{(5.8)}{\le} \frac{q_m n x_n + r_m x_1}{m}, \qquad \forall m \ge n \ge 1. \end{aligned} \tag{5.9}$$

For each fixed n, we regard the left and right sides of (5.9) as the m-th terms of two sequences. Although the inequality (5.9) is only valid from the n-th term onwards, $\overline{\lim}_{m\to\infty}$ does not depend on the first $n-1$ terms. So we have

$$\begin{aligned} \forall n \ge 1, \qquad \overline{\lim_{m\to\infty}}\, x_m &\le \overline{\lim_{m\to\infty}} \frac{q_m n x_n + r_m x_1}{m} && \text{(Exercise 4.7)} \\ &\le x_n \cdot \overline{\lim_{m\to\infty}} \frac{q_m n}{m} + x_1 \cdot \overline{\lim_{m\to\infty}} \frac{r_m}{m}. && \text{(Exercise 5.1)} \end{aligned} \tag{5.10}$$

In fact, the lim-sups on the right side of (5.10) are genuine limits, because

$$\lim_{m\to\infty} \frac{q_m n}{m} = \lim_{m\to\infty} \frac{m - r_m}{m} = 1 - \lim_{m\to\infty} \frac{r_m}{m} = 1,$$

and

$$\frac{0}{m} \le \frac{r_m}{m} \le \frac{n-1}{m} \overset{\text{squeezing}}{\Rightarrow} \lim_{m\to\infty} \frac{r_m}{m} = 0.$$

So (5.10) simplifies to

$$\forall n \ge 1, \qquad \overline{\lim_{m\to\infty}}\, x_m \le x_n.$$

Now regard the left side as the n-th term of a constant sequence, then take $\underline{\lim}_{n\to\infty}$ on both sides and use Exercise 4.7 to get

$$\overline{\lim_{m\to\infty}}\, x_m \le \underline{\lim_{n\to\infty}}\, x_n.$$

The reverse inequality automatically holds (Eq. (4.25)), so the lim-inf and lim-sup coincide. By Prop. 4.8, the sequence $x_1, x_2, \ldots$ is convergent.

- The above examples are rather artificial. Some situations where lim-sup and lim-inf arise naturally are:
  - Root test for convergence of infinite series, and power series (Analysis II course).
  - Borel–Cantelli lemma (Probability).
  - Measure theory and spectral theory.

  These are topics for future courses.

There is a close relationship between the study of sequences in $\mathbb{R}$, and the study of the *topology* of $\mathbb{R}$. Roughly speaking, "topology" refers to concepts of "nearby", thus it underlies the concepts of "limit" and "continuity". In future courses, you may study "abstract topological spaces", and limits/continuity in this context. For this course, we will only study the concrete topology of $\mathbb{R}$ associated with its natural distance function $\mathrm{dist}(x, y) = |x - y|$.

Below we will prove the following chain of implications:

$$\begin{array}{ccc} \text{Monotone convergence theorem} & \Longrightarrow & \text{Nested interval theorem} \\ & & \Downarrow \\ & & \text{Heine} - \text{Borel theorem} \\ & & \Downarrow \\ \text{Bolzano} - \text{Weierstrass theorem} & \Longleftarrow & \text{Limit point principle} \end{array}$$

The left side involves sequences, while the right side involves "topological" concepts, to be introduced shortly. On the left side, we had already derived BWT from MCT. As an Exercise, derive MCT from BWT (thus "closing the loop" in the above diagram).

We start with a brief discussion on the topology of intervals.

- *Bounded intervals* are subsets of $\mathbb{R}$ specified by real numbers $a < b$,

$$\begin{aligned} (a, b) &:= \{x \in \mathbb{R} \,:\, a < x < b\}, \\ (a, b] &:= \{x \in \mathbb{R} \,:\, a < x \leq b\}, \\ [a, b) &:= \{x \in \mathbb{R} \,:\, a \leq x < b\}, \\ [a, b] &:= \{x \in \mathbb{R} \,:\, a \leq x \leq b\}. \end{aligned}$$

The first case is called an *open* interval, while the last case is called a *closed* interval.

Unbounded intervals are of the form

$$\begin{aligned}(a, +\infty) &:= \{x \in \mathbb{R} : a < x\},\\ (-\infty, a) &:= \{x \in \mathbb{R} : x < a\},\\ (-\infty, \infty) &:= \mathbb{R},\\ [a, +\infty) &:= \{x \in \mathbb{R} : a \le x\},\\ (-\infty, a] &:= \{x \in \mathbb{R} : x \le a\},\end{aligned}$$

where a is a real number. The first three intervals are *open*, while the last three intervals are *closed*. (So $\mathbb{R}$ is both open and closed.)

Theorem 5.2 (Nested Interval Theorem)**.** *Let $[a_n, b_n], n \ge 1$ be a sequence of closed bounded intervals such that*

- $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$ *for all* $n \ge 1$*;*
- $\lim\limits_{n\to\infty} (b_n - a_n) = 0.$

Then $\bigcap\limits_{n\ge1} [a_n, b_n]$ *contains exactly one point.*

Proof. By the first assumption, the sequence $a_1, a_2, \ldots$ is monotone increasing and has an upper bound. By MCT, $\lim\limits_{n\to\infty} a_n$ exists. Together with the second assumption, we also have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n + \underbrace{\lim_{n\to\infty} (b_n - a_n)}_{=0} = \lim_{n\to\infty} a_n.$$

Let η be any element of $\bigcap_{n\ge1}[a_n, b_n]$, i.e.,

$$a_n \le \eta \le b_n, \qquad \forall n \ge 1.$$

By equality of the limits of the left and right sides, the squeezing principle forces $\eta = \lim\limits_{n\to\infty} a_n$.

It remains to verify that

$$\lim_{n\to\infty} a_n \in \bigcap_{n\ge1} [a_n, b_n]. \tag{5.11}$$

Since

$$a_n \le a_m < b_m \le b_n, \qquad \forall m \ge n \ge 1,$$

upon taking the $m \to \infty$ limit, we get

$$a_n \le \lim_{m\to\infty} a_m = \lim_{n\to\infty} b_m \le b_n, \qquad \forall n \ge 1.$$

Thus (5.11) holds. □

- To state the next theorem, we need the general concept of open/closed subsets of $\mathbb{R}$.

  Let E be a subset of $\mathbb{R}$.

  - A point $x_0 \in \mathbb{R}$ is an *interior point* of E, if

    $$\exists \delta > 0 \text{ such that } (x_0 - \delta, x_0 + \delta) \subseteq E. \tag{5.12}$$

    In this case, we also say that E is a *neighbourhood* of x_0, and that $E \setminus \{x_0\}$ is a *deleted neighbourhood* of x_0.
  - If every point of E is an interior point of E, we say that E is an *open set*.

    (Open intervals are open sets in this sense.)
  - If $\mathbb{R} \setminus E$ is open, we say that E is *closed*.

    (Closed intervals are closed sets in this sense.)
  - An *open cover* of E is a (possibly infinite) collection $\mathscr{F}$ of open sets such that

    $$E \subseteq \bigcup_{V\in\mathscr{F}} V.$$

    If a subcollection $\mathscr{F}' \subset \mathscr{F}$ is still an open cover of E, i.e.,

    $$E \subseteq \bigcup_{V\in\mathscr{F}'} V$$

    holds, we say that $\mathscr{F}'$ is a *subcover* of $\mathscr{F}$ (for E).

  Exercise 5.2. The collection

  $$\mathscr{F} = \{(n, n + \tfrac{3}{2}) : n \in \mathbb{Z}\}$$

  is an open cover of the *unbounded* set $\mathbb{R}$.

The collection

$$\mathscr{F} = \left\{ \left(\frac{1}{n+2}, \frac{1}{n} \right) \,:\, n \geq 1 \right\}$$

is an open cover of the *open, bounded* interval $(0, 1)$.

- For these two examples of open covers, are there any finite subcovers?
- Regard the second $\mathscr{F}$ as an open cover of the *closed, bounded* interval $[\frac{1}{3}, \frac{2}{3}]$. Find a finite subcover of $\mathscr{F}$.

Theorem 5.3 (Heine–Borel Theorem)**.** *Let E be a closed and bounded*[7] *subset of $\mathbb{R}$. Then every open cover of E has a finite subcover.*

Proof. As E is bounded, we may write $E \subseteq [a_1, b_1]$ for some $[a_1, b_1]$. Let $\mathscr{F}$ be an open cover of E. Note that $\mathscr{F}$ may not be an open cover of E. However, E is closed, so $\mathbb{R} \setminus E$ is open. Thus the enlarged collection

$$\widetilde{\mathscr{F}} = \mathscr{F} \cup \{\mathbb{R} \setminus E\}$$

is an open cover of $\mathbb{R}$, thus it covers $[a_1, b_1]$ (and also E).

Assume that $\widetilde{\mathscr{F}}$ has a finite subcover $\widetilde{\mathscr{F}'}$ (for the interval $[a_1, b_1]$). Then $\widetilde{\mathscr{F}'}$ also covers E. In fact,

$$\mathscr{F}' = \widetilde{\mathscr{F}'} \setminus \{\mathbb{R} \setminus E\}$$

covers E as well, since $\mathbb{R} \setminus E$ is redundant. So we obtain a finite subcover $\mathscr{F}' \subset \mathscr{F}$ (for E).

Therefore, *to prove Theorem 5.3 for a general closed, bounded $E \subseteq \mathbb{R}$, it suffices to prove it for the interval case, $E = [a_1, b_1]$.* We will do a proof-by-contradiction.

Assume Theorem 5.3 is false for the interval $E = [a_1, b_1]$. So there exists an open cover $\mathscr{F}$ of $[a_1, b_1]$, such that

$$\text{no finite subcollection of } \mathscr{F} \text{covers } [a_1, b_1]. \tag{5.13}$$

Write

$$[a_1, b_1] = \left[a_1, \frac{a_1 + b_1}{2} \right] \cup \left[\frac{a_1 + b_1}{2}, b_1 \right].$$

[7] A subset of $\mathbb{R}$ is *bounded* if it has an upper bound and a lower bound.

Define a subinterval

$$[a_2, b_2] := \begin{cases} \left[a_1, \frac{a_1+b_1}{2}\right], & \text{if no finite subcollection of } \mathscr{F} \text{ covers } \left[a_1, \frac{a_1+b_1}{2}\right], \\ \left[\frac{a_1+b_1}{2}, b_1\right], & \text{if some finite subcollection of } \mathscr{F} \text{ covers } \left[a_1, \frac{a_1+b_1}{2}\right]. \end{cases}$$

In the first case, no finite subcollection of $\mathscr{F}$ covers $[a_2, b_2] = \left[a_1, \frac{a_1+b_1}{2}\right]$. In the second case, no finite subcollection of $\mathscr{F}$ covers $[a_2, b_2] = \left[\frac{a_1+b_1}{2}, b_1\right]$; otherwise, we could combine a finite subcollection of $\mathscr{F}$ covering $\left[a_1, \frac{a_1+b_1}{2}\right]$ with a finite subcollection of $\mathscr{F}$ covering $\left[\frac{a_1+b_1}{2}, b_1\right]$, to obtain a finite subcollection of $\mathscr{F}$ covering of $[a_1, b_1]$, contradicting (5.13).

Similarly, write

$$[a_2, b_2] = \left[a_2, \frac{a_2+b_2}{2}\right] \cup \left[\frac{a_2+b_2}{2}, b_2\right],$$

and repeat the same argument to obtain a subinterval $[a_3, b_3] \subseteq [a_2, b_2]$ with half the length, with the property that no finite subcollection of $\mathscr{F}$ covers $[a_3, b_3]$.

Inductively, we obtain a sequence of closed intervals $[a_n, b_n], n \geq 1$, such that every $[a_n, b_n]$ has the property that no finite subcollection of $\mathscr{F}$ covers it. Furthermore, $b_n - a_n = \frac{b_1-a_1}{2^{n-1}}$, so the conditions of the Nested Interval Theorem 5.2 are satisfied. Therefore, $\bigcap\limits_{n\geq 1} [a_n, b_n]$ contains exactly one point, call it x_0. As $\mathscr{F}$ covers $[a_1, b_1]$, and $x_0 \in [a_1, b_1]$, there exists some $V \in \mathscr{F}$ such that $x_0 \in V$. As V is an open set, x_0 is an interior point of V, so there exists some $\delta > 0$ such that

$$(x_0 - \delta, x_0 + \delta) \subseteq V.$$

Because $\lim\limits_{n\to\infty} (b_n - a_n) = 0$, there exists some $N \geq 1$ such that $b_N - a_N < \delta$. As $x_0 \in [a_N, b_N]$, this implies that

$$[a_N, b_N] \subseteq (x_0 - \delta, x_0 + \delta) \subseteq V.$$

So $\{V\}$ covers $[a_N, b_N]$. But this contradicts the property that no finite subcollection of $\mathscr{F}$ covers $[a_N, b_N]$. □

- Let E be a subset of $\mathbb{R}$. We say that $x_0 \in \mathbb{R}$ is a *limit point* of E if

$$\forall \delta > 0, \qquad E \cap (x_0 - \delta, x_0 + \delta) \text{ has infinitely many points.} \tag{5.14}$$

The idea is that a limit point can be viewed as the limit of some sequence in E.

Note a limit point of E does not need to be an element of E; e.g., 0 is a limit point of $(0, 1]$.

Corollary 5.4 (Limit point principle)**.** *Any bounded infinite subset of* $\mathbb{R}$ *has a limit point.*

Proof. Let $E \subseteq \mathbb{R}$ be a bounded subset. We need to prove the implication

$$E \text{ infinite } \Rightarrow \ E \text{ has a limit point.}$$

We will prove the equivalent contrapositive implication,

$$E \text{ has no limit point } \Rightarrow \ E \text{ finite.}$$

Suppose E has no limit points. As E is bounded, we may write $E \subseteq [a, b]$ for some closed, bounded interval. By assumption, each $x \in [a, b]$ is not a limit point of E. So, by definition (5.14), there exists $\delta_x > 0$ such that

$$E \cap (x - \delta_x, x + \delta_x) \text{ is finite.} \tag{5.15}$$

The following collection of open intervals

$$\{(x - \delta_x, x + \delta_x) \ : \ x \in [a, b]\}$$

covers the closed bounded set $[a, b]$. By the Heine–Borel Theorem 5.3, there is a finite subcollection, say,

$$\big\{(x_1 - \delta_{x_1}, x_1 + \delta_{x_1}), (x_2 - \delta_{x_2}, x_2 + \delta_{x_2}), \ldots, (x_n - \delta_{x_n}, x_n + \delta_{x_n})\big\},$$

which covers $[a, b]$, thus it also covers E. So

$$E = E \cap \bigcup_{i=1}^{n} (x_i - \delta_{x_i}, x_i + \delta_{x_i}) = \bigcup_{i=1}^{n} E \cap (x_i - \delta_{x_i}, x_i + \delta_{x_i}). \tag{5.16}$$

Combining (5.15) and (5.16), E must be a finite set. □

Remark. Some authors refer to Corollary 5.4 as the "Bolzano–Weierstrass Theorem". This is because we can easily prove our Bolzano–Weierstrass Theorem 4.7 using Corollary 5.4. (Exercise, or see the proof of Theorem 3.3.10 of Lou–Yang–Mei book.)

By the way, we can also use the Bolzano–Weierstrass Theorem 4.7 to prove the Cauchy completeness Theorem 3.8 (Exercise, or see the proof of 3.3.12 of Lou–Yang–Mei book).

Limits and continuity of functions

In classical mechanics, one studies motion, e.g., how position depends on time. Here, space is conceived as a geometric "continuum" with no "gaps", and is presumed to be specifiable by some $\mathbb{R}$-valued position coordinates (measured by a ruler). For simplicity, we consider just a single position coordinate. Likewise, time is labelled by elements of $\mathbb{R}$ (measured by a clock). So motion is modelled and recorded by a real-valued function

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ t &\mapsto f(t). \end{aligned}$$

In the late 17-th century, Newton developed his calculus, to give powerful predictive "laws of motion" governing f. By today's standards, his calculus was not put on a rigorous foundation, but it was still able to achieve tremendous scientific applications and stimulated a lot of mathematical development.

The functions f used in calculus should at least be "continuous": Informally, as t approaches t_0, the values $f(t)$ should also approach $f(t_0)$. A rigorous definition of "limit", "continuity", "derivative", etc., was only achieved in the late 19th century!

Below, we will rigorously study the "limit" and "continuity" concepts for real-valued functions

$$f : \mathrm{Dom}(f) \to \mathbb{R},$$

where $\mathrm{Dom}(f) \subseteq \mathbb{R}$ refers to the *domain* of f. For example, the function $f : x \mapsto x^{-1}$ is usually understood to have $\mathrm{Dom}(f) = \mathbb{R} \setminus \{0\}$. Sometimes we do not care about the whole domain. *By convention, when we say "f is defined on some set E", we mean that $E \subseteq \mathrm{Dom}(f)$.*

- We start with a sequential formulation of limits and continuity.

  Fix $x > 0$. Recall the meaning of $f(y) = x^y$ for irrational exponents y. We started with a definition of $f(q) = x^q$ for rational q. Then, we observed that the same limit $L = \lim_{n\to\infty} f(q_n)$ is obtained, no matter which rational approach $q_1, q_2, \ldots$ was taken to reach y. This allowed us to define $f(y) = x^y$ to be that common limit. In fact, using this definition of $f(y)$, we can show that $\lim_{n\to\infty} f(y_n) = L$ holds for any *real* sequence $y_1, y_2, \ldots$ converging to y, see Example 6.1 later. What we have done is to extend $f : \mathbb{Q} \to \mathbb{R}$ to a function $f : \mathbb{R} \to \mathbb{R}$ which is *continuous* at every $y \in \mathbb{R}$, in the sense of Definition 3 below.

Definition 3 (Sequential definition of limit of function)**.** Let $x_0 \in \mathbb{R}$, and f be a real-valued function.

- Suppose f is defined on some deleted neighbourhood V of x_0. We say that f *has a limit* $L \in \mathbb{R}$ *at* x_0, if

$$\forall \text{ sequences } x_1, x_2, \dots \text{ in } V \text{ converging to } x_0, \quad \lim_{n\to\infty} f(x_n) = L \text{ holds.} \tag{5.17}$$

  In this case, we write $\lim\limits_{x\to x_0} f(x) = L$.

- Suppose f is defined on $(x_0 - \delta_0, x_0)$ for some $\delta_0 > 0$. We say that f has *left-sided limit* $L \in \mathbb{R}$ *at* x_0, if

$$\forall \text{ sequences } x_1, x_2, \dots \text{ in } (x_0 - \delta_0, x_0) \text{ converging to } x_0, \quad \lim_{n\to\infty} f(x_n) = L \text{ holds.}$$

  In this case, we write $\lim\limits_{x\to x_0^-} f(x) = L$.

- Suppose f is defined on $(x_0, x_0 + \delta_0)$ for some $\delta_0 > 0$. We say that f has *right-sided limit* $L \in \mathbb{R}$ *at* x_0, if

$$\forall \text{ sequences } x_1, x_2, \dots \text{ in } (x_0, x_0 + \delta_0) \text{ converging to } x_0, \quad \lim_{n\to\infty} f(x_n) = L \text{ holds.}$$

  In this case, we write $\lim\limits_{x\to x_0^+} f(x) = L$.

Definition 4 (Sequential continuity of function)**.** Let $x_0 \in \mathbb{R}$, and f be a real-valued function.

- Suppose f is defined on some neighbourhood W of x_0. We say that f is *continuous at* x_0, if

$$\lim_{x\to x_0} f(x) = f(x_0). \tag{5.18}$$

- Suppose f is defined on $(x_0 - \delta_0, x_0]$ for some $\delta_0 > 0$. We say that f is *left-continuous at* x_0 if

$$\lim_{x\to x_0^-} f(x) = f(x_0).$$

– Suppose f is defined on $[x_0, x_0 + \delta_0)$ for some $\delta_0 > 0$. We say that f is *right-continuous at* x_0 if

$$\lim_{x \to x_0^+} f(x) = f(x_0).$$

Some comments:

– You should think of each sequence $x_1, x_2, \ldots$ in V converging to x_0 as describing one possible "approach" to x_0. For f to have a limit at x_0, we require $f(x_n)$ to approach the same value, independently of the approach to x_0.

– The existence of the limit of f at x_0 depends on the values of the function f *near* x_0. It does *not* depend on the value of f *at* x_0; in fact, we do not even require $x_0 \in \mathrm{Dom}(f)$.

  Continuity of f at x_0 is just the extra requirement that $f(x_0)$ is defined and equals $\lim_{x \to x_0} f(x)$.

  When $\lim_{x \to x_0} f(x) = L$ exists but $x_0 \notin \mathrm{Dom}(f)$, we can extend the domain of f to include x_0, by setting $f(x_0) := L$. This extended f will be continuous at x_0, since (5.18) is satisfied by definition.

– The L such that (5.17) is satisfied is unique, because of the uniqueness of sequential limits.

– Suppose (5.17) holds. By definition (5.12) of neighbourhood,

$$\exists \delta_0 > 0 \text{ such that } (x_0 - \delta_0, x_0) \cup (x_0, x_0 + \delta_0) \subseteq V. \tag{5.19}$$

  After deleting finitely many initial terms, any sequence in V that converges to x_0 will still be a sequence in (5.19) converging to x_0. Thus, we may assume that $V = (x_0 - \delta_0, x_0) \cup (x_0, x_0 + \delta_0)$ for some $\delta_0 > 0$. Likewise, we can assume that $W = (x - \delta_0, x + \delta_0)$ for some $\delta_0 > 0$.

  In fact, by further deleting finitely many initial terms, we may reduce $\delta_0 > 0$ to any smaller $\delta_0' > 0$ in (5.19). (Analogously, the limit of a sequence is unaffected by modifying or deleting finitely many terms of the sequence.)

  Thus, the continuity (respectively, limit) of f at x_0 only depends on f in an *arbitrarily small* (respectively, deleted) neighbourhood of x_0. The specific choice of V, W in Definition 3 is not important.

– In terms of sequences, an equivalent definition of "f is continuous at x_0" is:

$$\begin{gathered}\forall \text{ sequences } x_1, x_2, \dots \text{ in } W \text{ converging to } x_0, \\ \lim_{n\to\infty} f(x_n) = f\Big(\lim_{n\to\infty} x_n\Big).\end{gathered} \tag{5.20}$$

Informally: Continuity of f at $x_0 \Leftrightarrow$ "$f(\cdot)$ commutes with $\lim_{n\to\infty}$".

Exercise: In (5.20), we might expect that we should only use sequences in the *deleted* neighbourhood $W \setminus \{x_0\}$ that converge to x_0. Why is it OK for some terms x_n to be x_0?

•

Exercise 5.3. Show that f has limit L at x_0 iff f has left-sided limit L at x_0 and right-sided limit L at x_0.

Deduce that f is continuous at x_0 iff f is left-continuous and right-continuous at x_0.

• Here are some simple examples.

Example 5.2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $x \mapsto x^2 + 1$. Fix any $x_0 \in \mathbb{R}$. If $x_1, x_2, \dots$ is any sequence in $\mathbb{R} \setminus \{x_0\}$ converging to x_0, then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (x_n^2 + 1) = \Big(\lim_{n\to\infty} x_n\Big)^2 + 1 = x_0^2 + 1$$

holds by algebraic properties of sequence limits. Thus

$$\lim_{x\to x_0} f(x) = x_0^2 + 1 = f(x_0), \qquad \forall\, x_0 \in \mathbb{R},$$

i.e., f is continuous at every $x_0 \in \mathbb{R}$.

Similarly, we deduce that polynomial functions are continuous at every $x_0 \in \mathbb{R}$.

Example 5.3. Consider the function

$$\begin{aligned} f : \mathbb{R} \setminus \{1\} &\to \mathbb{R} \\ x &\mapsto \frac{x^2 - 1}{x - 1}. \end{aligned}$$

We can write $f(x) = x + 1$ for all $x \in \mathbb{R} \setminus \{1\}$. If $x_1, x_2, \ldots$ is a sequence in $\mathbb{R} \setminus \{1\}$ converging to 1, then clearly

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (x_n + 1) = \left(\lim_{n\to\infty} x_n\right) + 1 = 1 + 1 = 2.$$

So $\lim_{x\to 1} f(x) = 2$.

We can extend f to the function

$$\begin{aligned} \tilde{f} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x + 1, \end{aligned}$$

by setting $\tilde{f}(1) = 2$. Then $\lim_{x\to 1} \tilde{f}(x) = 2 = \tilde{f}(1)$ holds, so $\tilde{f}$ is continuous at 1.

Exercise 5.4. Show that the function

$$\begin{aligned} f : \mathbb{R} \setminus \{0\} &\to \mathbb{R} \\ x &\mapsto \frac{1}{x}, \end{aligned}$$

does not satisfy $\lim_{x\to 0} f(x) = L$ for any $L \in \mathbb{R}$. (This implies that there is no way to extend f to a function which is continuous at 0.)

Exercise 5.5. The *Dirichlet* function is

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases} \end{aligned}$$

For every $x_0 \in \mathbb{R}$, show that f does not have a limit at x_0. (Thus it is nowhere continuous.)

- Algebraic properties of limits of functions follow quite easily from Definition 3 and the corresponding algebraic properties of limits of sequences.

  Exercise 5.6. Let f, g be real-valued functions defined on some deleted neighbourhood V of a point $x_0 \in \mathbb{R}$. Suppose the limits

  $$\lim_{x\to x_0} f(x) = L, \qquad \lim_{x\to x_0} g(x) = M$$

  exist.

(i) Show that for any $\lambda, \mu \in \mathbb{R}$, we have

$$\lim_{x \to x_0} (\lambda f + \mu g)(x) = \lambda L + \mu M.$$

(ii) Show that $\lim_{x \to x_0} fg(x) = LM$.

(iii) If $M \neq 0$, show that

$$\lim_{x \to x_0} \left(\frac{f}{g}\right)(x) = \frac{L}{M}.$$

Notes: $\lambda f + \mu g$ is the function whose value at x is $\lambda f(x) + \mu g(x)$. Similarly, fg is the function whose value at x is $f(x)g(x)$.

By definition, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, unless $g(x) = 0$ (then it is not defined there). The assumption that $\lim_{x \to x_0} g(x) = M \neq 0$ implies that $g(x) \neq 0$ holds on some possibly smaller deleted neighbourhood V' of x_0 (why?). For the purposes of investigating the limit of $\frac{f}{g}$ at x_0, we can assume that it is defined on such a deleted neighbourhood V'.

- We also mention some consequences of BWT for limits of *bounded* functions.

  Exercise 5.7. Let f be a real-valued function, which is bounded on some deleted neighbourhood V of x_0.

  - Show that there exists a sequence $x_1, x_2, \ldots$ in V converging to x_0, such that the sequence $f(x_1), f(x_2), \ldots$ converges.
  - Assume that there exists $L \in \mathbb{R}$ such that for all sequences $x_1, x_2, \ldots$ converging to x_0,

    $$f(x_1), f(x_2), \ldots \text{ converges } \Rightarrow \lim_{n \to \infty} f(x_n) = L.$$

    Show that

    $$\lim_{x \to x_0} f(x) = L.$$