6 Lectures 11–12

- There is also a squeezing principle for limits of functions.

  Proposition 6.1 (Squeezing principle)**.** *Let f, g, h be real-valued functions, and assume that*

  $$f(x) \leq g(x) \leq h(x), \qquad \forall\, x \in V, \tag{6.1}$$

  holds on some deleted neighbourhood V of $x_0 \in \mathbb{R}$. If the limits of f and h at x_0 exist and are equal, then

  $$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x).$$

  The same principle holds for one-sided limits.

  Proof. As f, h are assumed to have the same limit at x_0, all sequences $x_1, x_2, \ldots$ in V that converge to x_0 satisfy

  $$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} h(x_n).$$

  The assumption (6.1) also says that

  $$f(x_n) \leq g(x_n) \leq h(x_n), \qquad \forall n \geq 1,$$

  so by the squeezing principle for sequences,

  $$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} h(x_n).$$

  As the sequence $x_1, x_2, \ldots$ in V converging to x_0 was arbitrarily chosen, this shows that the function g also has the same limit as f, h at x_0.

  The same squeezing argument applies for one-sided limits. □

- Recall the sequential limit definition of Euler's number, Eq. (4.20),

  $$e := \lim_{n \to \infty} (1 + \frac{1}{n})^n. \tag{6.2}$$

  Below, we will use the squeezing principle to derive a related function limit formula,

  $$e = \lim_{x \to 0} (1 + x)^{1/x}. \tag{6.3}$$

Notation: for $a \in \mathbb{R}$, we write $\lfloor a \rfloor$ for the largest integer which is smaller than or equal to a, so

$$\lfloor a \rfloor \le a < \lfloor a \rfloor + 1.$$

For $x > 0$, it follows from (4.9), (4.10) that

$$\left(1 + \frac{1}{\lfloor x^{-1} \rfloor + 1}\right)^{\lfloor x^{-1} \rfloor} < \overbrace{\left(1 + \frac{1}{x^{-1}}\right)^{x^{-1}}}^{(1+x)^{1/x}} < \left(1 + \frac{1}{\lfloor x^{-1} \rfloor}\right)^{\lfloor x^{-1} \rfloor + 1}. \quad (6.4)$$

We first show that the $x \to 0^+$ limit of the left and right sides of (6.4) are both equal to e. Let $x_1, x_2, \ldots$ be any sequence in $(0, 1)$ which converges to 0. Let $\epsilon > 0$ be given. The limit formula (6.2) says that

$$\exists M \ge 1,\ \forall m \ge M, \qquad \left|(1 + \frac{1}{m})^m - e\right| < \epsilon. \quad (6.5)$$

For this M, the convergence $x_n \to 0$ says that

$$\begin{aligned} \exists N,\ \forall n \ge N, \qquad & 0 < x_n < \frac{1}{M+1} \\ & \Rightarrow x_n^{-1} > M + 1 \\ & \Rightarrow \lfloor x_n^{-1} \rfloor > M \\ & \Rightarrow \begin{cases} \left|\left(1 + \frac{1}{\lfloor x_n^{-1} \rfloor}\right)^{\lfloor x_n^{-1} \rfloor} - e\right| < \epsilon \\ \left|\left(1 + \frac{1}{\lfloor x_n^{-1} \rfloor + 1}\right)^{\lfloor x_n^{-1} \rfloor + 1} - e\right| < \epsilon \end{cases} \qquad \text{(by Eq. (6.5))} \end{aligned} \quad (6.6)$$

– [Right side of (6.4)]. We have

$$\lim_{n\to\infty} \left(1 + \frac{1}{\lfloor x_n^{-1} \rfloor}\right)^{\lfloor x_n^{-1} \rfloor + 1} = \underbrace{\lim_{n\to\infty} \left(1 + \frac{1}{\lfloor x_n^{-1} \rfloor}\right)^{\lfloor x_n^{-1} \rfloor}}_{=e \text{ by first case of (6.6)}} \cdot \underbrace{\lim_{n\to\infty} \left(1 + \frac{1}{\lfloor x_n^{-1} \rfloor}\right)}_{=1} = e.$$

– [Left side of (6.4)]. Similarly, we have

$$\lim_{n\to\infty} \left(1 + \frac{1}{\lfloor x_n^{-1} \rfloor + 1}\right)^{\lfloor x_n^{-1} \rfloor} = \underbrace{\lim_{n\to\infty} \left(1 + \frac{1}{\lfloor x_n^{-1} \rfloor + 1}\right)^{\lfloor x_n^{-1} \rfloor + 1}}_{=e \text{ by second case of (6.6)}} \Big/ \underbrace{\lim_{n\to\infty} \left(1 + \frac{1}{\lfloor x_n^{-1} \rfloor + 1}\right)}_{=1} = e.$$

By the squeezing principle, the middle function in (6.4) also has $x \to 0^+$ limit equal to e,

$$\lim_{x\to 0^+} (1+x)^{1/x} = e. \tag{6.7}$$

As for the left-sided limit at 0, we have

$$\lim_{x\to 0^-} (1+x)^{1/x} = \lim_{x\to 0^+} (1-x)^{-1/x} = \lim_{x\to 0^+} (1+\frac{x}{1-x})^{1/x}$$

Check that $x_n \to 0$ iff $x'_n := \frac{x_n}{1-x_n} \to 0$. So we may change variables to $x' = \frac{x}{1-x}$ (or $\frac{1}{x} = \frac{1}{x'} + 1$), to obtain

$$\lim_{x\to 0^-} (1+x)^{1/x} = \lim_{x'\to 0^+} (1+x')^{1/x'+1} = \lim_{x'\to 0^+} (1+x')^{1/x'} \cdot 1 \overset{(6.7)}{=} e. \tag{6.8}$$

Eq. (6.7)–(6.8), together with Exercise 5.3, verify the claim (6.3).

- Let us use our *analytic* tools to establish the continuity of some *non-algebraically* defined functions.

  Example 6.1. For fixed $x > 0$, the exponential function is

  $$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ y &\mapsto x^y. \end{aligned}$$

  It will suffice to show that f is continuous at $y_0 = 0$. Using this fact, we can deduce that at a general $y_0 \in \mathbb{R}$,

  $$\begin{aligned} \lim_{y\to y_0} f(y) = \lim_{y\to y_0} x^y = \lim_{y\to y_0} x^{y_0} f(y-y_0) &= x^{y_0} \lim_{y\to y_0} f(y-y_0) \quad \text{(Exercise 5.6)} \\ &= x^{y_0} \lim_{y'\to 0} f(y') \qquad (y' := y - y_0) \\ &= x^{y_0} \cdot 1 = f(y_0), \end{aligned}$$

  so f is continuous at $y_0 \in \mathbb{R}$. (For the change of variables, note that $y_n \to y_0$ iff $y'_n := y_n - y_0 \to 0$.)

  So, let $y_1, y_2, \ldots$ be a real sequence converging to 0. Let $\epsilon > 0$ be given. Recall from Exercise 4.2 that $\lim_{n\to\infty} x^{1/n} = 1$. By the quotient rule, we also have $\lim_{n\to\infty} x^{-1/n} = 1$.

  So there exists a natural number $N \geq 1$ such that

  $$|x^{1/N} - 1| < \epsilon \qquad \text{and} \qquad |x^{-1/N} - 1| < \epsilon. \tag{6.9}$$

As the sequence $y_1, y_2, \ldots$ converges to 0,

$$\exists M \geq 1,\ \forall n \geq M, \qquad -1/N < y_n < 1/N. \tag{6.10}$$

In the case $x > 1$, we use the monotonicity (4.10), together with (6.9), (6.10), to deduce that

$$\exists M \geq 1,\ \forall n \geq M, \qquad |x^{y_n} - 1| = \begin{cases} x^{y_n} - 1 < x^{1/N} - 1 < \epsilon, & \text{if } y_n \geq 0, \\ 1 - x^{y_n} < 1 - x^{-1/N} < \epsilon, & \text{if } y_n \leq 0. \end{cases} \tag{6.11}$$

Similarly, inequality (6.11) holds for the case $0 < x < 1$; the $x = 1$ case is straightforward.

We have thus shown, for any $x > 0$, that for any sequence $y_1, y_2, \ldots$ converging to 0,

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} x^{y_n} = 1 = x^0 = f(0) = f\left(\lim_{n\to\infty} y_n\right)$$

holds. This means that f is continuous at 0 (see the formulation (5.20)).

Example 6.2. Fix $y \in \mathbb{R}$. We show that the power function

$$\begin{aligned} f : (0, \infty) &\to \mathbb{R} \\ x &\mapsto x^y \end{aligned}$$

is continuous on $(0, \infty)$.

We first check continuity at $x_0 = 1$; equivalently, by Exercise 5.3, we check left-continuity and right-continuity at $x_0 = 1$. There are three cases: $y > 0, y = 0, y < 0$. We only spell out the $y > 0$ case.

For right-continuity at 1: Let $x_1, x_2, \ldots$ be any sequence in $(1, 2)$ converging to 1. Fix a positive integer N such that $N \geq y$. By algebraic properties of limits, we have

$$\lim_{n\to\infty} x_n^N = \left(\lim_{n\to\infty} x_n\right)^N = 1^N = 1.$$

By monotonicity, we also have $1 \leq x_n^y \leq x_n^N$, so the squeezing principle gives

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n^y = 1 = f(1).$$

Therefore, f is right-continuous at 1.

For left-continuity at 1: Similar squeezing argument (with reversed monotonicity).

For continuity at a general $x_0 \in (0, \infty)$, change variables to $\tilde{x} = \frac{x}{x_0}$, so

$$x^y = x_0^y \cdot \left(\frac{x}{x_0}\right)^y = x_0^y \cdot \tilde{x}^y$$

A sequence $x_1, x_2, \ldots$ converges to x_0 iff $\tilde{x}_1, \tilde{x}_2, \ldots$ converges to $\frac{x_0}{x_0} = 1$. Then

$$\lim_{x \to x_0} x^y = x_0^y \lim_{x \to x_0} \left(\frac{x}{x_0}\right)^y = x_0^y \lim_{x \to x_0} f(x/x_0) = x_0^y \lim_{\tilde{x} \to 1} f(\tilde{x}) = x_0^y,$$

where the last equality uses continuity of f at 1 that we had established earlier. So

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x^y = x_0^y = f(x_0),$$

which is the definition of continuity of f at x_0.

- Subsequently, we will often study functions defined on some fixed subset $E \subseteq \mathbb{R}$, not necessarily a (deleted) neighbourhood, and we will be interested in limit/continuity at various points $x_0 \in E$. So, *it will be convenient to use a slight generalization of Definition 3.*

  Definition 5 (Sequential limit/continuity)**.** Let $E \subseteq \mathbb{R}$.

  – If x_0 is a limit point of E, we say that $f : E \setminus \{x_0\} \to \mathbb{R}$ *has limit* L *at* x_0, if

  $$\forall \text{ sequences } x_1, x_2, \ldots \text{ in } E \setminus \{x_0\} \text{ converging to } x_0, \quad \lim_{n \to \infty} f(x_n) = L \text{ holds.} \tag{6.12}$$

  – We say that $f : E \to \mathbb{R}$ *is continuous at* $x_0 \in E$, if

  $$\forall \text{ sequences } x_1, x_2, \ldots \text{ in } E \text{ converging to } x_0, \ \lim_{n \to \infty} f(x_n) = f(x_0). \tag{6.13}$$

  We say that f is *continuous on* E, if it is continuous at every $x_0 \in E$.

  Remarks:

– If $E \setminus \{x_0\} = V$ is a deleted neighbourhood of x_0, then Definition 5 of limit at x_0 coincides with the older Definition 3. In this case, x_0 is automatically a limit point of E.

  Similarly, if $E = W$ is neighbourhood of x_0, then Definition 5 of continuity at x_0 coincides with the older Definition 4 (compare Eq. (6.13) with Eq. (5.20)).

– Definition 5 of continuity at x_0 makes sense, whether or not x_0 is a limit point of E.

  If x_0 happens to be a limit point of E, then f is continuous at x_0 iff f has limit $f(x_0)$ at x_0.

– The point 0 is an isolated point of $E = \{0\} \cup [1, 2]$. In this example, 0 is *not* a limit point of $E \setminus \{0\}$ (recall definition (5.14)). So we do not ask[9] the question "Does $f : E \setminus \{0\} \to \mathbb{R}$ have a limit at 0?". But we can ask "Is $f : E \to \mathbb{R}$ continuous at 0?".

  A sequence $x_1, x_2, \ldots$ in $E = \{0\} \cup [1, 2]$ converges to 0 iff there exists some $N \geq 1$ such that $x_n = 0$ for all $n \geq N$ (why?). For such a sequence, the sequence of values, $f(x_1), f(x_2), \ldots$, has a constant tail with value $f(0)$. Therefore, the continuity condition (6.13) at the isolated point 0 is automatically satisfied by *any* $f : E \to \mathbb{R}$.

– If $E = [a, b]$, then continuity of $f : E \to \mathbb{R}$ at a or b (Definition 5) means right-continuity at a or left-continuity at b (Definition 4). Similarly, limits at a or b (Definition 5) mean one-sided limits (Definition 3).

  This makes Definition 5 convenient for discussing continuity on a whole interval.

– For continuity (or limit) at a single point $x_0 \in E$, we may consider f to be defined only on $E \cap (x_0 - \delta_0, x_0 + \delta_0)$ (or $E \cap ((x_0 - \delta_0, x_0) \cup (x_0, x_0 + \delta_0))$). Any choice of $\delta_0 > 0$ may be used, and such a restriction does not change the continuity (or limit) property of f at x_0.

  However, we cannot simply choose to consider f to be defined on some *arbitrary* subset $E' \subset E$. For example, if $E = (-1, 1)$, and $E' = \{0\}$, then any $f : E \to \mathbb{R}$ would automatically become continuous when regarded as $f : \{0\} \to \mathbb{R}$.

[9] Unfortunately, different authors have different viewpoints on whether the question of limits at isolated points makes sense.

– Recall that limits of convergent sequences in E are limit points of $E \subseteq \mathbb{R}$. Conversely:

Exercise 6.1. Show that a limit point x_0 of a subset $E \subseteq \mathbb{R}$ arises as the limit of some sequence in $E \setminus \{x_0\}$. (This requires axiom of countable choice, discussed below Theorem 6.5.)

So (under axiom of countable choice), the defining condition (6.12) of limit is not an empty condition — there is at least one convergent sequence to check.

Note (not examinable): Any conclusion in an empty condition is true. So, if there are no sequences converging to x_0 at all, the statement "for all sequences in $E\setminus\{x_0\}$ converging to x_0, $\lim_{n\to\infty} f(x_n) = L$" would be considered true, no matter what L is. So Definition 5 of limit would be rather problematic if we did not allow axiom of choice.

- We shall investigate continuity of functions defined on intervals. Intervals in the complete ordered field $\mathbb{R}$ were described in the discussion above Theorem 5.2. The are nine types[10].

  We can characterize intervals in $\mathbb{R}$ abstractly, by a "connectedness" property:

  Proposition 6.2 (Characterization of intervals)**.** *A subset $E \subseteq \mathbb{R}$ is an interval iff it contains at least two points, and satisfies the property*

  $$\forall x, y \in E, \qquad x < y \;\Rightarrow\; [x, y] \subseteq E. \tag{6.14}$$

  Proof. ($\Rightarrow$) is immediate from the definitions of the various types of intervals.

  ($\Leftarrow$). We use the completeness property, on a case-by-case basis.

  – [E bounded]. We have

  $$E \subseteq [a, b], \qquad a = \inf E, b = \sup E, \tag{6.15}$$

  where $a \neq b$ because E has at least two distinct points. If $c \in (a, b)$, then it is not an upper/lower bound for E. So there exists $a', b' \in E$

[10] Sometimes, $\emptyset$ and $\{a\}$ are also considered to be intervals, but we do not include them.

such that $c \in (a', b')$. By assumption (6.14), we have $[a', b'] \subseteq E$, so $c \in E$. As $c \in (a, b)$ was arbitrary, we have shown that

$$(a, b) \subseteq E. \tag{6.16}$$

Combining (6.15), (6.16), we see that E is in between (a, b) and $[a, b]$, so it must be one of the four bounded intervals with endpoints a, b.

– [E unbounded]. This is left as an exercise.

□

Proposition 6.3 (Preservation of intervals)**.** *Let I be an interval and $f : I \to \mathbb{R}$ be continuous on I and non-constant. Then its range $f(I)$ is an interval.*

Proof. Let $f(a) < f(b)$ be any two points in $f(I)$. We may assume that $a < b$ (the $b < a$ case is handled similarly). Pick a point $y_0 \in (f(a), f(b))$. We claim that

$$\forall y_0 \in (f(a), f(b)),\ \exists x_0 \in (a, b) \text{ such that } y_0 = f(x_0). \tag{6.17}$$

Thus $[f(a), f(b)] \subseteq f(I)$. As $f(a) < f(b)$ were arbitrarily chosen from $f(I)$, Property (6.14) of Prop. 6.2 is satisfied, thus $f(I)$ is an interval.

It remains to verify the claim (6.17). Let $y_0 \in (f(a), f(b))$.

– Set $a_1 = a, b_1 = b$, so

$$f(a_1) < y_0 < f(b_1).$$

– The midpoint $\frac{a_1+b_1}{2}$ lies in I. If $f(\frac{a_1+b_1}{2}) = y_0$, we are done. Otherwise, define a_2, b_2 by

$$\begin{cases} a_2 = a_1, \quad b_2 = \frac{a_1+b_1}{2}, & \text{if } f(\frac{a_1+b_1}{2}) > y_0, \\ a_2 = \frac{a_1+b_1}{2}, b_2 = b_1, & \text{if } f(\frac{a_1+b_1}{2}) < y_0, \end{cases}$$

so that

$$f(a_2) < y_0 < f(b_2).$$

Again, the midpoint $\frac{a_2+b_2}{2}$ lies in I. If $f(\frac{a_2+b_2}{2}) = y_0$, we are done. Otherwise, define a_3, b_3 similarly.

– If after a finite number of interval-halving steps, we obtain $f(\frac{a_n+b_n}{2}) = y_0$, we are done. Otherwise, we will have constructed two sequences in I satisfying

$$f(a_n) < y_0 < f(b_n), \qquad n \geq 1, \tag{6.18}$$

and

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots b_n \leq \ldots \leq b_2 \leq b_1.$$

By MCT, $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist, and we also have

$$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty} (b_n - a_n) = 0$$

So $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$. Call this common limit x_0. Note that x_0 lies in $[a_1, b_1]$, so it lies in the domain I of f.

– Use continuity of f to obtain

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(x_0) = f\left(\lim_{n\to\infty} b_n\right) = \lim_{n\to\infty} b_n.$$

Putting this limit equality into (6.18) and applying the squeezing principle gives the claim (6.17).

□

Theorem 6.4 (Bolzano's Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then for all y strictly between $f(a)$ and $f(b)$, there exists $x \in (a, b)$ such that $f(x) = y$.*

Proof. This is basically a special case of the Preservation of Intervals Proposition 6.3, but with conclusion which is slightly more specific than "range of f is an interval"[11].

That Preservation of Intervals Proposition guarantees that $f[a, b]$ is an interval; furthermore, we have f defined at the end points a, b, so $f(a), f(b) \in$

[11] If I is an interval which does not contain both end-points, then the "initial value $f(a)$" and "final value $f(b)$" are not defined. It does not make sense to write an IVT statement in the specific form in Theorem 6.4. Instead, the analogue of Theorem 6.4 for continuous f on general intervals is precisely the Preservation of Intervals Proposition 6.3, saying that "range of f is an interval". As an Exercise, use the IVT and the Characterization of Intervals Proposition to prove the Preservation of Intervals Proposition.

$f[a,b]$. If $f(a) = f(b)$, there is nothing to check. Otherwise, assume without loss of generality that $f(a) < f(b)$. Then $[f(a), f(b)] \subseteq f[a,b]$. In particular, any $y \in \big(f(a), f(b)\big)$ lies in $f[a,b]$, i.e., $y = f(x)$ for some $x \in [a,b]$. As $f(a) \neq y \neq f(b)$, we cannot have $x = a$ or $x = b$, so we must actually have $x \in (a,b)$. □

Remark. There are many different ways to prove the IVT, Theorem 6.4. We chose a fairly minimal approach, which uses only completeness (and its fairly straightforward consequence, the MCT).

Exercise: Prove the completeness property from the IVT.

- Let $E \subseteq \mathbb{R}$. If x is the limit of a sequence in $E \setminus \{x\}$, then it is a limit point of E. More generally, if x is the limit of a sequence in E, then either x is a limit point of E, or $x \in E$. (The second case is due to sequences whose tails constantly x.)

  In general, a limit point of $E \subseteq \mathbb{R}$ may not belong to E. However, we claim that:

$$\text{For closed } E \subseteq \mathbb{R}, \quad x \text{ is a limit point of } E \;\Rightarrow\; x \in E. \tag{6.19}$$

  We may prove the equivalent contrapositive implication as follows: Let $x \in \mathbb{R} \setminus E$. Since $\mathbb{R} \setminus E$ is open, there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq \mathbb{R} \setminus E$. Then $E \cap (x-\delta, x+\delta)$ is empty, so x is not a limit point of E.

  From the above discussion, we obtain the useful rule:

$$\text{For closed } E \subseteq \mathbb{R}, \quad x \text{ is a limit of a sequence in } E \;\Rightarrow\; x \in E. \tag{6.20}$$

Theorem 6.5 (Extreme Value Theorem/Maximum principle)**.** *Let E be a closed and bounded subset of $\mathbb{R}$. If a real-valued function f is continuous on E, then there exist $a, b \in E$ such that*

$$f(a) = \inf f(E), \qquad f(b) = \sup f(E).$$

Proof. The BWT for bounded sequences, as well as the rule (6.20) for closed subsets, are used in both steps below.

  - [$f(E)$ is bounded]. Suppose $f(E)$ has no upper bound. So, for each $n \geq 1$, there exists $x_n \in E$ such that $f(x_n) > n$. The sequence $x_1, x_2, \ldots$ in E is bounded, so BWT gives a convergent subsequence

$x_{n_1}, x_{n_2}, \ldots$ in E. Denote its limit by x_0, which lies in E because E is closed (Eq. (6.20)).

By assumption, f is continuous at x_0, so the sequence $f(x_{n_1}), f(x_{n_2}), \ldots$ should converge to $f(x_0)$. But $f(x_{n_1}), f(x_{n_2}), \ldots$ is an *unbounded* sequence by construction; contradiction.

So $f(E)$ has an upper bound. Similarly, it has a lower bound. Thus (by completeness), the infimum and supremum of $f(E)$ exist in $\mathbb{R}$.

– [Inf/sup of $f(E)$ are attained]. Write $M = \sup f(E)$. For each $n \geq 1$, the number $M - \frac{1}{n}$ is not an upper bound for $f(E)$. So we can find $x_n \in E$ such that

$$M - \frac{1}{n} < f(x_n) \leq M, \qquad n \geq 1.$$

Note that $f(x_1), f(x_2), \ldots$ converges to M by the squeezing principle.

The sequence $x_1, x_2, \ldots$ lies in the bounded set E, so BWT gives a subsequence $x_{n_1}, x_{n_2}, \ldots$ converging to some limit b. As E is closed, $b \in E$. Using continuity of f at x_0,

$$f(b) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = M = \sup f(E).$$

Similarly, we can find $a \in E$ such that $f(a) = \inf f(E)$.

□

Remark (Not examinable). In the proof of Theorem 6.5, we had to make *infinitely* many choices (one choice for each $n \geq 1$) to obtain a sequence $x_1, x_2, \ldots$. Before this part of the lectures, we had been able to avoid performing such an "infinite choice". For example, in the proof of BWT, the monotone subsequence provided by the preparatory Lemma 4.6 can be recursively specified by taking the smallest admissible label n_k at each step k. (The subset of admissible labels is a subset of $\mathbb{N}$, and we use the well-ordering principle to specify the smallest one.)

In fact, one needs to use an "Axiom of countable choice" to justify making $|\mathbb{N}|$-many *choices*, in the absence of a recursive *formula*. In this course, we freely assume the commonly-used ZFC axioms, where the "C" refers to an even stronger "Axiom of Choice" allowing for arbitrary (not just countable-many) choices.

Interestingly, if we restrict the statement of the EVT to closed bounded intervals $E = [a, b]$, it is possible to give a recursive formula specifying the desired sequences $x_1, x_2, \ldots$ in the proof. So the choice axiom can be avoided in this case.

ϵ-δ Formulation

- Here is an alternative definition of limit/continuity of a function.

  Definition 6 (ϵ-δ definition of limit/continuity of function)**.** Let $E \subseteq \mathbb{R}$.

  - Let x_0 be a limit point of E, and $f : E \setminus \{x_0\} \to \mathbb{R}$. We say that f has *limit* $L \in \mathbb{R}$ *at* $x_0 \in E$, if

    $$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in E, \quad 0 < |x - x_0| < \delta \ \Rightarrow\ |f(x) - L| < \epsilon. \tag{6.21}$$

  - Let $f : E \to \mathbb{R}$. We say that f is *continuous at* $x_0 \in E$, if

    $$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in E, \qquad |x - x_0| < \delta \ \Rightarrow\ |f(x) - f(x_0)| < \epsilon. \tag{6.22}$$

  Remarks:

  - In Theorem 6.6, we will show that the sequential and the ϵ-δ definitions of limit/continuity are equivalent.
  - If condition (6.21) holds for some $\delta > 0$, it also holds for any smaller choice of δ. So the existence of the limit of f at x_0 (Definition 6) depends only on the values of f arbitrarily near to x_0.

    Similarly for continuity at x_0. (But we need to know the value $f(x_0)$ in this case.)
  - The δ making (6.21) or (6.22) true will generally depend on the given ϵ. So we sometimes write $\delta \equiv \delta(\epsilon)$. It also depends on x_0, and we will discuss this dependence later, when we introduce the concept of "uniform continuity".

    The $\epsilon - \delta$ Definition 6 of limit can be interpreted in terms of "controlled approximation": Suppose we are given an error bound ϵ. We can ensure that the values of f stay within the ϵ error bound from the limit L, by working over parameters x that are within $\delta(\epsilon)$ distance from x_0.

If we fail to control x, i.e., x is allowed to deviate a lot from x_0, then we cannot guarantee ϵ-control of errors for f. In applications (engineering, physics, economics, ... and their modelling), error control is very important, and this is why a rigorous understanding of limit/continuity properties of f is important.

Theorem 6.6 (Heine)**.** *Definitions 5 and 6 are equivalent.*

Proof. Let $E \subseteq \mathbb{R}$, x_0 be a limit point of E, and $f : E \setminus \{x_0\} \to \mathbb{R}$,

- Suppose f has limit L at x_0 in the ϵ-δ sense. Let $x_1, x_2, \ldots$ be any sequence in $E \setminus \{x_0\}$ converging to x_0, and let $\epsilon > 0$ be given. There exists $\delta > 0$ such that

$$\forall x \in E, \quad 0 < |x - x_0| < \delta \;\Rightarrow\; |f(x) - L| < \epsilon. \tag{6.23}$$

  For this $\delta > 0$, the convergence $\lim_{n\to\infty} x_n = x_0$ ensures that

$$\exists N \geq 1,\ \forall n \geq N, \qquad |x_n - x_0| < \delta \overset{(6.23)}{\Rightarrow} |f(x_n) - L| < \epsilon$$

  Therefore, $\lim_{n\to\infty} f(x_n) = L$ holds. This verifies that f has limit L at x_0 in the sequential sense.

- Suppose f does *not* have limit L at x_0 in the ϵ-δ sense. Then there exists an $\epsilon > 0$ such that

$$\forall \delta > 0,\ \exists x \in E \text{ with } 0 < |x - x_0| < \delta \text{ such that } |f(x) - L| \geq \epsilon.$$

  For each $n \geq 1$, setting $\delta = \frac{1}{n}$ in the above statement, we can choose[12] an $x_n \in E \setminus \{x_0\}$ satisfying

$$x_0 - \frac{1}{n} < x_n < x_0 + \frac{1}{n} \quad \text{and} \quad |f(x_n) - L| \geq \epsilon.$$

  Due to the squeezing principle,

$$\lim_{n\to\infty} x_n = x_0.$$

  However, this sequence $x_1, x_2, \ldots$ in $E \setminus \{x_0\}$ also satisfies

$$|f(x_n) - L| \geq \epsilon, \qquad \forall n \geq 1.$$

  So $f(x_1), f(x_2), \ldots, f(x_n), \ldots$ cannot converge to L. Thus f fails the sequential criterion (6.12) to have limit L at x_0.

[12] Axiom of countable choice is needed here. We cannot avoid this!

Similarly, to show that the two definitions of continuity at x_0 are equivalent, replace "sequence in $E \setminus \{x_0\}$" by "sequence in E", L by $f(x_0)$, and $0 < |x - x_0| < \delta$ by $|x - x_0| < \delta$ in the above argument. □

- For some practice, show that the functions in Examples 5.2–5.3 have the stated limits in the sense of the ϵ-δ definition. Also, show that the function in Exercise 5.4 does not have a limit at 0 in the ϵ-δ sense. You may also try to re-prove the algebraic properties in Exercise 5.6 using the ϵ-δ definition of limit.

  Exercise 6.2. Show that the function

  $$\begin{aligned} f : (0, \infty) &\to \mathbb{R} \\ x &\mapsto \sqrt{x}, \end{aligned}$$

  is continuous on $x_0 \in (0, \infty)$, using the ϵ-δ definition of continuity.

  (It should be a little easier than the general case, Example 6.2, which used the sequential definition.)

- Here are some other basic properties of limits of functions, which you can prove using the ϵ-δ definitions. (More generally, you could replace V by $E \setminus \{x_0\}$ below.)

  Exercise 6.3. Let f, g be real-valued functions defined on some deleted neighbourhood V of a point $x_0 \in \mathbb{R}$. Suppose the limits

  $$\lim_{x \to x_0} f(x) = L, \qquad \lim_{x \to x_0} g(x) = M$$

  exist.

  (i) Suppose $L > 0$. Show that there exists some $\delta > 0$ such that $f(x) > 0$ holds for all $x \in V$ with $0 < |x - x_0| < \delta$.

  (ii) Suppose there exists $\delta > 0$ such that

  $$f(x) \leq g(x), \qquad \forall x \in V \text{ with } 0 < |x - x_0| < \delta.$$

  Show that $L \leq M$.

  (iii) Show that there exists $\delta > 0$ such that the set

  $$\{f(x) : x \in V,\ |x - x_0| < \delta\}$$

  is bounded.