7 Lectures 13–14

- The following "algebraic stability" of continuity is easily deduced from Exercise 5.6.

  Exercise 7.1. Let $E \subseteq \mathbb{R}$, and $f, g : E \to \mathbb{R}$ be real-valued functions which are continuous at a point $x_0 \in E$. Show that:d

  (i) For any $\lambda, \mu \in \mathbb{R}$, the function $\lambda f + \mu g$ is continuous at x_0.

  (ii) The function fg is continuous at x_0.

  (iii) If $g(x_0) \neq 0$, then the function $\frac{f}{g}$ is continuous at x_0.

  You should also try to do Exercise 7.1 using the ϵ-δ-definition of continuity. (For (iii), you might need to use the fact that there exists $c > 0$ such that given $\epsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |g(x)| > c > 0$; why?)

  In general, for two real-valued functions f, g, their *composition*

  $$g \circ f : x \mapsto g(f(x))$$

  has domain comprising those $x \in \mathrm{Dom}(f)$ such that $f(x) \in \mathrm{Dom}(g)$.

  Proposition 7.1. *Suppose $f : E \to \mathbb{R}$ is continuous at $x_0 \in E$, $g : E' \to \mathbb{R}$ is continuous at $f(x_0) \in E'$, and*[13] *$f(E) \subseteq E'$. Then $g \circ f : E \to \mathbb{R}$ is continuous at x_0.*

  Proof. Let $\epsilon > 0$ be given. By ϵ-δ' definition of continuity of g at $f(x_0)$, there exists $\delta' > 0$ such that

  $$\forall y \in E', \qquad |y - f(x_0)| < \delta' \;\Rightarrow\; |g(y) - g(f(x_0))| < \epsilon. \tag{7.1}$$

  By δ'-δ definition of continuity of f at x_0, there exists $\delta > 0$ such that

  $$\forall x \in E, \qquad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \delta' \overset{f(E)\subseteq E' \text{ and } (7.1)}{\Rightarrow} |g(f(x)) - g(f(x_0))| < \epsilon.$$

  This shows that $g \circ f$ is continuous at x_0. □

[13] In the neighbourhood setting, if f is defined on some neighbourhood of x_0 and g is defined on some neighbourhood of $f(x_0)$, we can show that continuity of g at $f(x_0)$ implies that $g \circ f$ is also defined on some neighbourhood of x_0. In other words, we can arrange for $f(E) \subseteq E'$ to hold by choosing appropriate neighbourhoods. In the general setting, we should initially specify $f(E) \subseteq E'$.

- Intuitively, a function $f : I \to \mathbb{R}$ whose graph can be drawn without lifting the pen, is continuous on I. Yet, it is possible for f to be continuous only at a *single* point:

  Exercise 7.2. Show that

  $$f : \mathbb{R} \to \mathbb{R}$$
  $$x \mapsto \begin{cases} x, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

  is continuous at 0 and discontinuous elsewhere.

  (On $\mathbb{R} \setminus \{0\}$, relate f to the Dirichlet function and use Prop. 7.1.)

  So pointwise continuity can be quite counter-intuitive to visualize.

- A sequence in $\mathbb{R}$ has a limit iff it has the Cauchy property. Likewise, there is a "Cauchy criterion" for existence of limits of functions.

  Proposition 7.2. *Let $E \subseteq \mathbb{R}$ and x_0 be a limit point of E. Then $f : E \setminus \{x_0\} \to \mathbb{R}$ has a limit at x_0 iff*

  $\forall \epsilon > 0,\ \exists \delta > 0$ such that
  $\forall x, x' \in E,\ \ 0 < |x - x_0| < \delta$ and $0 < |x' - x_0| < \delta \ \Rightarrow\ |f(x) - f(x')| < \epsilon.$ (7.2)

  Proof. By Theorem 6.6, we may use either the sequential definition or the ϵ-δ definition of limit.

  ($\Leftarrow$) Given $\epsilon > 0$, we pick some $\delta > 0$ such that (7.2) holds. Let $x_1, x_2, \ldots$ be a sequence in $E \setminus \{x_0\}$ converging to x_0. So, for the above $\delta > 0$, there exists some $N \geq 1$ such that

  $$0 < |x_m - x_0| < \delta \ \text{ and } \ 0 < |x_n - x_0| < \delta, \qquad \forall m, n \geq N,$$

  then (7.2) applies, giving

  $$|f(x_m) - f(x_n)| < \epsilon, \qquad \forall m, n \geq N.$$

  We have verified the Cauchy condition for the sequence $f(x_1), f(x_2), \ldots$ of values. Thus $\lim_{n\to\infty} f(x_n) = L$ for $L \in \mathbb{R}$.

If we had used some other sequence $\tilde{x}_1, \tilde{x_2}, \ldots$ in $E \setminus \{x_0\}$ converging to x_0, we would have obtained $\lim_{n\to\infty} f(\tilde{x}_n) = \tilde{L}$ for some $\tilde{L} \in \mathbb{R}$. We could regard the x_n and $\tilde{x}_n$ sequences as even/odd subsequences of a single sequence in $E \setminus \{x_0\}$. The L and $\tilde{L}$ would be the limits of some subsequences of the same convergent sequence, thus $L = \tilde{L}$.

We have verified that *any* sequence $x_1, x_2, \ldots$ in $E \setminus \{x_0\}$ converging to x_0 satisfies $\lim_{n\to\infty} f(x_n) = L$. So f has limit L at x_0 (sequential definition).

($\Rightarrow$) Suppose f has limit L at x_0 (ϵ-δ definition). So, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\forall x \in E, \qquad 0 < |x - x_0| < \delta \;\Rightarrow\; |f(x) - L| < \epsilon/2.$$

Combine with the triangle inequality,

$$\begin{aligned}\forall x, x' \in E, \quad & 0 < |x - x_0| < \delta \text{ and } 0 < |x' - x_0| < \delta \\ & \Rightarrow\; |f(x) - f(x')| \le |f(x) - L| + |L - f(x')| < \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

So condition (7.2) holds. □

If you prefer to use the ϵ-δ definition of limit in the $\Rightarrow$ direction above, do the following exercise:

Exercise 7.3. Use the Nested Interval Theorem to show: If condition (7.2) holds, then f has a limit at x_0 (ϵ-δ sense).

Exercise 7.4. Let $f : \mathbb{R} \to \mathbb{R}$ be the "zig-zag" function defined by $x \mapsto |x|$ when $x \in [-1, 1]$, and extended to a 2-periodic function, $f(x + 2) = f(x), \forall x \in \mathbb{R}$. Define the function

$$\begin{aligned} g : (0, \infty) &\to \mathbb{R}, \\ x &\mapsto f(\tfrac{1}{x}). \end{aligned}$$

Show that g does not have a limit at 0. (This means $\lim_{x\to 0^+} g(x)$ does not exist, in the terminology of Definition 3.)

Remark: g is an example of a function which oscillates wildly as $x \to 0$. Another standard textbook example is the function $x \mapsto \sin(\frac{1}{x})$, but we cannot handle this example rigorously, because we have not defined sin yet.

- Let us say that a real-valued function f is *monotone* on a subset $E \subseteq \mathrm{Dom}(f)$, if

$$\forall x, y \in E, \quad x \leq y \;\Rightarrow\; \begin{cases} f(x) \leq f(y), & \text{(increasing)} \\ f(x) \geq f(y). & \text{(decreasing).} \end{cases}$$

We say that it is *strictly monotone* if

$$\forall x, y \in E, \quad x < y \;\Rightarrow\; \begin{cases} f(x) < f(y), & \text{(increasing)} \\ f(x) > f(y). & \text{(decreasing).} \end{cases}$$

The following result is an analogue of MCT for functions on intervals, and may be derived from the completeness property.

Exercise 7.5. Let f be a monotone increasing function defined on an interval I. Show that if c is an interior point of I, then

$$\lim_{x \to c^+} f(x) = \inf_{x > c} f(x), \qquad \lim_{x \to c^-} f(x) = \sup_{x < c} f(x)$$

exist. If I contains its left endpoint a, then

$$\lim_{x \to a^+} f(x) = \inf_{x > a} f(x)$$

exists. If I contains its right endpoint b, then

$$\lim_{x \to a^-} f(x) = \sup_{x < a} f(x)$$

exists. Suppose f is monotone decreasing. Show that the above limits exist, but with inf and sup exchanged in the formulas.

If f is monotone on an interval with a point c removed, e.g., $(a, c) \cup (c, b)$, then Exercise 7.5 says that f has a left-sided limit and a right-sided limit at c. *But they need not be equal*, so f may not have a limit at c.

Let $f : I \to \mathbb{R}$ be a monotone function defined on an interval I. By Exercise 7.5, the following definitions make sense.

- For an interior point $c \in I$, we define the *jump of f at c* to be

$$j(c) := \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x). \tag{7.3}$$

– If I includes its left endpoint a, we define

$$j(a) := \lim_{x \to a^+} f(x) - f(a).$$

If I includes its right endpoint b, we define

$$j(b) := f(b) - \lim_{x \to b^-} f(x).$$

The following result follows immediately from the definition of $j(\cdot)$ and Exercise 5.3.

Lemma 7.3. *Let $f : I \to \mathbb{R}$ be a monotone function defined on an interval I. Then f is continuous at $c \in I$ iff $j(c) = 0$.*

- An injective function $f : \mathrm{Dom}(f) \to \mathbb{R}$ determines an inverse function $f : \mathrm{Range}(f) \to \mathbb{R}$. However, continuity of an injective f does not guarantee continuity of its inverse f^{-1}.

  Example 7.1. Consider the function

$$f : [0,1) \cup [2,3] \to \mathbb{R}$$
$$x \mapsto \begin{cases} x, & x \in [0,1), \\ x-1, & x \in [2,3]. \end{cases}$$

  This is continuous on its domain, and injective, with range $[0,2]$. The inverse function is

$$f^{-1} : [0,2] \to \mathbb{R}$$
$$y \mapsto \begin{cases} y, & y \in [0,1), \\ y+1, & y \in [1,2], \end{cases}$$

  which is discontinuous at 1, because of the jump there.

  The problem with Example 7.1 is the "disconnectedness" of its domain. The situation is better when f is defined on an interval.

Theorem 7.4. *If $f : I \to \mathbb{R}$ is* strictly *monotone and continuous on an interval I, then the inverse function f^{-1} is strictly monotone and continuous on $f(I)$.*

Proof. We assume that f is increasing; the decreasing case is similar.

First, strict monotonicity implies injectivity (why?), thus bijectivity of f from I to $f(I)$. So the inverse function

$$\begin{aligned} f^{-1} : f(I) &\to \mathbb{R} \\ f(x) &\mapsto x, \qquad\qquad x \in I, \end{aligned}$$

makes sense. Furthermore, is is straightforward to see that the strict monotone increasing property of f on I is equivalent to

$$\forall f(x), f(x') \in f(I), \qquad f(x) < f(x') \;\Rightarrow\; f^{-1}(f(x) = x < x' = f^{-1}(f(x')),$$

i.e., strict monotone increasing property of f^{-1} on $f(I)$.

Next, because f is continuous, the Preservation of Intervals proposition 6.3 implies that $f(I)$ is also an interval. Let $y_0 \in f(I)$, and suppose f^{-1} is *not* continuous at y_0.

– [y_0 is interior point of $f(I)$.] As $f(I)$ is an interval, we may use Lemma 7.3 (and Exercise 7.5) to deduce that f^{-1} has left-sided and right-sided limits at y_0, differing by a positive jump,

$$\underbrace{\sup_{y<y_0} f^{-1}(y)}_{= \lim\limits_{y \to y_0^-} f^{-1}(y)} < \underbrace{\inf_{y>y_0} f^{-1}(y)}_{= \lim\limits_{y \to y_0^+} f^{-1}(y)}.$$

Let us choose some number x_0 in the gap, distinct from $f^{-1}(y_0)$, i.e.,

$$\sup_{y<y_0} f^{-1}(y) < x_0 < \inf_{y<y_0} f^{-1}(y), \qquad x_0 \neq f^{-1}(y_0). \tag{7.4}$$

The inequalities (7.4) lead to a contradiction:

* $x_0 \neq f^{-1}(y)$ for all $y < y_0$ and all $y > y_0$, and also $y = y_0$. So $x_0 \neq f^{-1}(y)$ for all $y \in f(I)$. So $x_0 \notin I$.
* x_0 lies strictly between between some numbers $f^{-1}(y_1)$ and $f^{-1}(y_2)$ in I. Since I is an interval, we should have $x_0 \in I$.

– [y_0 is an end-point of $f(I)$.] Similar argument: Replace the left or right side of (7.4) by $f^{-1}(y_0)$.

Therefore, f^{-1} is continuous at y_0, for every $y_0 \in f(I)$. □

- In Example 7.1, we had a continuous injective function defined on the disjoint union of *two* intervals, but its inverse was discontinuous. Below, we will see why we cannot get such an example if its domain is a single interval.

  Lemma 7.5. *Let $f : I \to \mathbb{R}$ be a continuous function on an interval I. If f is not strictly monotone on I, then there exists $a < b < c$ in I such that*

  $$f(b) \geq \max\{f(a), f(c)\} \qquad \text{or} \qquad f(b) \leq \min\{f(a), f(c)\}. \tag{7.5}$$

  Proof. Intuitively, if f is not strictly monotone, then it must have a "peak" or a "valley" somewhere. Exercise: make this idea precise, to match it with the conclusion of the Lemma. □

  Corollary 7.6. *A continuous injective function $f : I \to \mathbb{R}$ on an interval I has a continuous inverse.*

  Proof. By Theorem 7.4, it is enough to argue that such a function f must be strictly monotone.

  Suppose otherwise, that f is *not* strictly monotone. Then by Lemma 7.5 and injectivity of f, there exist $a < b < c$ in I satisfying (7.5) with *strict* inequality.

  – [$f(b) > \max\{f(a), f(c)\}$ case.] Pick $\eta \in \big(\max\{f(a), f(c)\}, f(b)\big)$. By IVT applied to $\eta \in (f(a), f(b)$ and $\eta \in (f(c), f(b))$, there exist $\xi_1 \in (a, b)$ and $\xi_2 \in (b, c)$ such that

  $$f(\xi_1) = \eta = f(\xi_2).$$

  Note that ξ_1, ξ_2 are distinct. So the previous equation contradicts injectivity of f.

  – [$f(b) < \min\{f(a), f(c)\}$ case.] Pick $\eta \in \big(f(b), \min\{f(a), f(c)\}\big)$. By IVT, there exist *distinct* $\xi_1 \in (a, b)$ and $\xi_2 \in (b, c)$ such that

  $$f(\xi_1) = \eta = f(\xi_2),$$

  contradicting injectivity of f.

In either case, we get a contradiction, so f is strictly monotone. $\square$

- Here is an quick application of Theorem 7.4.

  Example 7.2. Let $x > 0, x \neq 1$, and consider the *logarithm* function

  $$\begin{aligned} g : (0, \infty) &\to \mathbb{R} \\ z &\mapsto \log_x z. \end{aligned}$$

  Here, recall from Exercise 4.5 that $\log_x z$ is defined as the unique real number such that

  $$x^{\log_x z} = z.$$

  In Example 6.1, we showed that the exponential function $f : y \mapsto x^y$ is continuous, and by Exercise 4.5, it has range $(0, \infty)$. In fact, g is the inverse function to f, by definition. We know from (4.10) and (4.11) that for $x > 0, x \neq 1$, the exponential function f is strictly monotone (decreasing if $0 < x < 1$, and increasing if $x > 1$). By Theorem 7.4, the logarithm function g is continuous on $(0, \infty)$.

  Exercise 7.6. Apply the continuity of $\ln := \log_e(\cdot)$ to (6.3) to show that

  $$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1. \tag{7.6}$$

  Deduce that

  1. For $x \in \mathbb{R}$, the sequential limit formula (4.20) for e generalizes to

     $$e^x = \lim_{n \to \infty} \big(1 + \frac{x}{n}\big)^n. \tag{7.7}$$

  2. $$\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

  3. For any $\alpha \in \mathbb{R}$,

     $$\lim_{x \to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha.$$

- Below, we will convert the sequential limit formula (7.7) for e^x into a *power series* formula,

  $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad x \in \mathbb{R}. \tag{7.8}$$

  Here, $k! := 1 \cdot 2 \cdots (k-1) \cdot k$ and $0! := 1$.

– First, check that the infinite sum (7.8) converges, for every $x \in \mathbb{R}$.
Exercise 7.7. Let $x \in \mathbb{R}$. Check that the following limit of partial sums exists:

$$\sum_{k=0}^{\infty} \frac{|x|^k}{k!} \equiv \lim_{m\to\infty} \sum_{k=0}^{m} \frac{|x|^k}{k!}.$$

Use (for example) the Cauchy criterion to deduce that

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \equiv \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x^k}{k!} \quad \text{exists.}$$

– Consider[14] the n-th term in the sequential formula (7.7) for e^x,

$$\begin{aligned}\big(1+\frac{x}{n}\big)^n = \sum_{k=0}^{n} \binom{n}{k} \Big(\frac{x}{n}\Big)^k 1^{n-k} &= \sum_{k=0}^{n} \frac{x^k n!}{n^k k!(n-k)!} \\ &= \sum_{k=0}^{n} \frac{x^k}{k!} \frac{n}{n} \frac{n-1}{n} \cdots \frac{n-k+1}{n}.\end{aligned}$$

Fix $m \geq 1$, then for all $n \geq m$, we have

$$\begin{aligned}\left|\big(1+\frac{x}{n}\big)^n - \sum_{k=0}^{m} \frac{x^k}{k!} \frac{n}{n} \frac{n-1}{n} \cdots \frac{n-k+1}{n}\right| &= \left|\sum_{k=m+1}^{n} \frac{x^k}{k!} \frac{n}{n} \frac{n-1}{n} \cdots \frac{n-k+1}{n}\right| \\ &\overset{\text{triangle}}{\leq} \sum_{k=m+1}^{n} \frac{|x|^k}{k!}\end{aligned}$$

by triangle inequality, thus

$$-\sum_{k=m+1}^{n} \frac{|x|^k}{k!} \leq \big(1+\frac{x}{n}\big)^n - \sum_{k=0}^{m} \frac{x^k}{k!} \frac{n}{n} \frac{n-1}{n} \cdots \frac{n-k+1}{n} \leq \sum_{k=m+1}^{n} \frac{|x|^k}{k!}.$$

Taking $n \to \infty$ and substituting (7.7),

$$-\sum_{k=m+1}^{\infty} \frac{|x|^k}{k!} \leq e^x - \sum_{k=0}^{m} \frac{x^k}{k!} \leq \sum_{k=m+1}^{\infty} \frac{|x|^k}{k!}, \qquad \forall m \geq 1. \tag{7.9}$$

[14]Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of ways to choose k objects out of n objects. This formula can be proved by induction, and we omit the proof.

– We have

$$\sum_{k=m+1}^{\infty} \frac{|x|^k}{k!} \equiv \lim_{n\to\infty} \sum_{k=m+1}^{n} \frac{|x|^k}{k!} = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{|x|^k}{k!} - \sum_{k=0}^{m} \frac{|x|^k}{k!} \right)$$
$$= \sum_{k=0}^{\infty} \frac{|x|^k}{k!} - \sum_{k=0}^{m} \frac{|x|^k}{k!}, \qquad \forall m \geq 1,$$

so

$$\lim_{m\to\infty} \sum_{k=m+1}^{\infty} \frac{|x|^k}{k!} = 0.$$

Therefore, the $m \to \infty$ limit of (7.9) and the squeezing principle give

$$e^x - \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x^k}{k!} = 0,$$

verifying the power series formula (7.8) for e^x.

Similarly to Exercise 7.7, the following power series are well-defined,

$$\cos x := \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \qquad \sin x := \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \qquad x \in \mathbb{R}. \quad (7.10)$$

These power series formulae define the functions $\cos : \mathbb{R} \to \mathbb{R}$ and $\sin : \mathbb{R} \to \mathbb{R}$.

- Let us prove the continuity of cos (the sin case is similar).

  For each fixed $x_0 \in \mathbb{R}$, given $\epsilon > 0$, we set $\delta = \min\{\epsilon e^{-(|x_0|+1)}, 1\}$. We compare the convergent infinite sums for $\cos x$ and $\cos x_0$,

$$\cos x - \cos x_0 = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k (x^{2k} - x_0^{2k})}{(2k)!}$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^k (x - x_0)(x^{2k-1} + x^{2k-2}x_0 + \ldots + x x_0^{2k-2} + x_0^{2k-1})}{(2k)!}$$
$$= (x - x_0) \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^k}{(2k)!} \sum_{j=0}^{2k-1} x^{2k-1-j} x_0^j.$$

For $|x-x_0|<\delta$, we have $|x|<|x_0|+1$. So, taking absolute values under the $\sum$ and using the triangle inequality[15], we get

$$\begin{aligned}
|x-x_0|<\delta \;\Rightarrow\; |\cos x-\cos x_0| &\le |x-x_0|\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(2k)(|x_0|+1)^{2k-1}}{(2k)!}\\
&= |x-x_0|\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(|x_0|+1)^{2k-1}}{(2k-1)!}\\
&< |x-x_0|\lim_{n\to\infty}\sum_{\ell=0}^{2n}\frac{(|x_0+1|)^{\ell}}{\ell!}\\
&< \delta e^{|x_0|+1}\\
&< \epsilon.
\end{aligned}$$

This shows that cos is continuous at every $x_0\in\mathbb{R}$.

Obviously, $\cos 0 = 1$ is positive. The following calculation shows that $\cos(2)$ is negative:

$$\begin{aligned}
\cos 2 &= 1-\frac{2^2}{2!}+\frac{2^4}{4!}+\lim_{n\to\infty}\left(-\frac{2^6}{6!}+\frac{2^8}{8!}-\ldots-\frac{2^{4n-2}}{(4n-2)!}+\frac{2^{4n}}{(4n)!}\right)\\
&= -\frac{1}{3}+\lim_{n\to\infty}\sum_{k=2}^{n}\left(-\frac{2^{4k-2}}{(4k-2)!}+\frac{2^{4k}}{(4k)!}\right)\\
&= -\frac{1}{3}-\lim_{n\to\infty}\sum_{k=2}^{n}\frac{1}{(4k-2)!}\Big(\underbrace{1-\frac{2^2}{4k(4k-1)}}_{>0}\Big) < -\frac{1}{3} < 0.
\end{aligned}$$

We know that cos is continuous on the interval $[0,2]$. Since $0\in(\cos(2),\cos(0))$, the Intermediate Value Theorem guarantees that there is some $x\in(0,2)$ such that $\cos x = 0$.

We have found that the bounded set

$$Z := \{x\in(0,2)\,:\,\cos(x)=0\}$$

is non-empty, so it has an infimum in $\mathbb{R}$. As $\cos(0)=1$, and cos is continuous, there exists some $\delta>0$ such $0<x<\delta \Rightarrow \cos(x)>\frac{1}{2}$

[15]Exercise: why is $\left|\sum_{k=0}^{\infty}a_k\right| \le \sum_{k=0}^{\infty}|a_k|$ valid when the right side converges?

(why?). So $\inf Z \geq \delta > 0$. Also, by a general property of infimum, either $\inf Z \in Z$, or $\inf Z$ is the limit of some sequence $z_1, z_2, \ldots$ in Z. In the second case, continuity of cos gives

$$\cos(\inf Z) = \cos\left(\lim_{n\to\infty} z_n\right) = \lim_{n\to\infty} \cos(z_0) = \lim_{n\to\infty} 0 = 0.$$

So $\inf Z \in Z$ in either case. One possible definition of π is: $\frac{\pi}{2}$ is the smallest positive solution to $\cos(x) = 0$, i.e.,

$$\pi = 2 \cdot \inf Z = 2 \cdot \min\{x > 0 \,:\, \cos(x) = 0\}.$$

Remark. Just like the Euclidean line has a natural parametrization using real numbers, the Euclidean plane has a natural parametrization using *complex* numbers. The plane geometry concepts of circles, triangles, trigonometry etc., are very elegantly described with complex number analysis. In particular, the famous $i = \sqrt{-1}$ is directly related to the new geometric feature of *rotations* of the plane, which is not a feature of the line.

The cosine and sine functions can be introduced geometrically, but *informally*, using the side lengths of right-angled triangles inscribed in a unit circle, with angle x. Lengths of line segments can be understood based on our existing constructions, but the meaning of "angle x" requires the concept of lengths of *arcs*, which requires calculus.

Another approach to $\cos, \sin$, is based on the fact that the power series (7.8) also converges for arbitrary *complex* numbers $x \in \mathbb{C}$. Some authors use (7.8) as the definition of e^x, then e would be defined as $\sum_{k=0}^{\infty} \frac{1}{k!}$. Then, by considering the formula

$$e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!}, \qquad x \in \mathbb{R},$$

and defining its the real and imaginary parts to be $\cos x$ and $\sin x$ respectively, i.e.,

$$e^{ix} = \cos x + i \sin x,$$

we obtain the formulae (7.10). But this approach is best explained after developing some analysis of complex numbers, and general power series theory, and these topics are only covered in the Analysis II course.

(Tao only introduces e, the complex exponential power series, and the $\sin, \cos$ functions in the second volume of his textbooks.)

Uniform continuity

- Let $E \subseteq \mathbb{R}$, and suppose $f : E \to \mathbb{R}$ is continuous at every $y \in E$. So the ϵ-δ criterion (6.22) for continuity is satisfied at every $y \in E$:

$$\forall y \in E,\ \forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in E, \qquad |x - y| < \delta \ \Rightarrow\ |f(x) - f(y)| < \epsilon. \tag{7.11}$$

  Above, the δ depends on both $y \in E$, and $\epsilon > 0$. We sometimes write $\delta(\epsilon; y)$ to emphasize this dependence.

  Definition 7. A real-valued function $f : E \to \mathbb{R}$ is said to be *uniformly continuous* on E, if

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x, y \in E, \qquad |x-y| < \delta \ \Rightarrow\ |f(x)-f(y)| < \epsilon. \tag{7.12}$$

  Compared to (7.11), the δ in (7.12) only depends on $\epsilon > 0$ (so we sometimes write $\delta(\epsilon)$. So uniform continuity, Eq. (7.12), is a stronger version of pointwise continuity at every $y \in E$, Eq. (7.11). Specifically, the *same* $\delta \equiv \delta(\epsilon)$ can be chosen to ensure

$$\forall x, y \in E, \qquad |x - y| < \delta \ \Rightarrow\ |f(x) - f(y)| < \epsilon.$$

  Remarks:

  - If $f : E \to \mathbb{R}$ is (uniformly) continuous on E, it remains (uniformly) continuous when regarded as $f|_{E'} : E' \to \mathbb{R}$ for any subset $E' \subseteq E$.
  - The concept of uniform continuity on intervals is important for Riemann's integration theory. This will be covered in Analysis II.

  Proposition 7.7. *A real-valued function $f : E \to \mathbb{R}$ is uniformly continuous on E iff for all pairs of sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ in E,*

$$\lim_{n\to\infty} (x_n - y_n) = 0 \ \Rightarrow\ \lim_{n\to\infty} (f(x_n) - f(y_n)) = 0. \tag{7.13}$$

  Proof. – ($\Rightarrow$). Let $\epsilon > 0$ be given. By the uniform continuity assumption (7.12) on f, there exists $\delta > 0$ such that (7.12) holds. Now consider any pair of sequences in E whose terms have differences converging to 0 (left side of Eq. (7.13)). Then, for the $\delta \equiv \delta(\epsilon) > 0$ mentioned above, there exists $N \equiv N(\delta(\epsilon)) \geq 1$ such that

$$\forall n \geq N, \qquad |x_n - y_n| < \delta \ \overset{(7.12)}{\Rightarrow}\ |f(x_n) - f(y_n)| < \epsilon.$$

  This verifies the right side of (7.13).

– ($\Leftarrow$). We prove the contrapositive statement: f is not uniformly continuous $\Rightarrow$ there exists a pair of sequences in E such that

$$\lim_{n\to\infty}(x_n - y_n) = 0 \quad \text{and} \quad f(x_1)-f(y_1), f(x_2)-f(y_2), \dots \text{ does not converge to } 0. \tag{7.14}$$

Failure of uniform continuity of f means

$$\exists \epsilon > 0,\ \forall \delta > 0,\ \exists x, y \in E, \qquad |x-y| < \delta \ \text{ and } \ |f(x)-f(y)| \geq \epsilon > 0.$$

For each $n \geq 1$, set $\delta = \frac{1}{n}$ in the above statement to get

$$\exists \epsilon > 0,\ \forall n \geq 1,\ \exists x_n, y_n \in E, \qquad |x_n-y_n| < \frac{1}{n} \ \text{ and } \ |f(x_n)-f(y_n)| \geq \epsilon > 0.$$

Inspecting the $n \to \infty$ limits, we have found two sequences[16] such that (7.14) holds.

□

Corollary 7.8. *If $f : E \to \mathbb{R}$ is uniformly continuous on E, then it maps Cauchy sequences in E to Cauchy sequences in $\mathbb{R}$.*

Proof. Exercise. □

Exercise 7.8. Let $f : E \to \mathbb{R}$ be uniformly continuous on E. Show that for any bounded subset $E' \subseteq E$, the set $f(E')$ is also bounded.

Example 7.3. By Exercise 7.1, the function

$$\begin{aligned} f : (0,\infty) &\to \mathbb{R} \\ x &\mapsto x^{-1} \end{aligned}$$

is continuous at every $x_0 \in (0,\infty)$. Consider the sequences

$$\frac{1}{1}, \frac{1}{2}, \dots, \frac{1}{n}, \dots, \qquad \text{and} \qquad \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2n}, \dots,$$

in $(0,\infty)$. The sequence of their differences is

$$\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2n}, \dots,$$

[16] Axiom of countable choice is used here.

which converges to 0. However, applying f then taking the difference gives the divergent sequence

$$-1, -2, \ldots, -n, \ldots.$$

According to the sequential definition, Prop. 7.7, the function f is not uniformly continuous on $(0, \infty)$.

In fact, the same argument shows that for any $a > 0$, the function $f : x \mapsto x^{-1}$ is *not* uniformly continuous on $(0, a]$.

- If $X \subseteq \mathbb{R}$, the *diameter* of X is

$$\operatorname{diam} X : \sup\{|x - y| \,:\, x, y \in X\}.$$

If $\mathscr{F}$ is an open cover of E, a *Lebesgue's number* of $\mathscr{F}$ is a number $\Delta > 0$ such that

$$\forall X \subseteq E, \qquad \operatorname{diam} X < \Delta \;\Rightarrow\; \exists V \in \mathscr{F} \text{ such that } X \subseteq V.$$

Lemma 7.9 (Lebesgue Covering Lemma)**.** *Let E be a closed and bounded subset of $\mathbb{R}$. Then any open cover of E has a Lebesgue's number.*

Proof. Let $\mathscr{F}$ be an open cover of E. Each $x \in E$ is an interior point of some open set $V \in \mathscr{F}$, so there exists some open interval $(x - r, x + r)$ contained in that V. Define

$$\mathscr{F}_0 := \big\{(x-r, x+r) \,:\, x \in E, r > 0, \exists V \in \mathscr{F} \text{ such that } (x-2r, x+2r) \in V\big\}.$$

Then $\mathscr{F}_0$ is an open cover of E. As E is closed and bounded, the Heine–Borel Theorem 5.3 says that there exists some finite subcover of $\mathscr{F}_0$. Fix such a finite subcover. It is specified by finitely many $x_k \in E, r_k > 0$, $V_k \in \mathscr{F}$, satisfying

$$(x_k - 2r_k, x_k + 2r_k) \subseteq V_k, \qquad 1 \le k \le n, \tag{7.15}$$

and

$$\mathscr{F}_0' = \big\{(x_k - r_k, x_k + r_k) \,:\, 1 \le k \le n\big\} \text{ covers } E. \tag{7.16}$$

Set

$$\Delta = \min_{1 \le k \le n} r_k > 0. \tag{7.17}$$

Given any (non-empty) $X \subseteq E$ with $\operatorname{diam} X < \Delta$, pick any $x_0 \in X$. By the covering property (7.16), x_0 lies in some $(x_k - r_k, x_k + r_k)$. Then by the triangle inequality,

$$X \subseteq [x_0 - \Delta, x_0 + \Delta] \subseteq (x_k - r_k - \Delta, x_k + r_k + \Delta) \overset{(7.17)}{\subseteq} (x_k - 2r_k, x_k + 2r_k) \overset{(7.15)}{\subseteq} V_k \in \mathscr{F}.$$

This shows that Δ is a Lebesgue's number for $\mathscr{F}$. □

Theorem 7.10 ((Heine–Cantor) Uniform Continuity theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then it is uniformly continuous on $[a, b]$.*

Proof. Let $\epsilon > 0$ be given. By continuity at each $x_0 \in [a, b]$,

$$\exists \delta(\tfrac{\epsilon}{2}, x_0) > 0,\ \forall x \in [a, b], \qquad |x - x_0| < \delta \ \Rightarrow\ |f(x) - f(x')| < \tfrac{\epsilon}{2}. \quad (7.18)$$

So the following collection is an open cover of $[a, b]$,

$$\big\{\big(x_0 - \delta(\tfrac{\epsilon}{2}, x_0), x_0 + \delta(\tfrac{\epsilon}{2}, x_0)\big) \,:\, x_0 \in [a, b]\big\}.$$

By the Lebesgue Covering Lemma 7.9, this cover has a Lebesgue's number $\Delta \equiv \Delta(\epsilon) > 0$.

For all pairs $x, y \in [a, b]$ satisfying $|x - y| < \Delta(\epsilon)$, we deduce that

$$\begin{aligned}
&\operatorname{diam}\{x, y\} < \Delta \\
&\Rightarrow \exists x_0 \in [a, b],\ \ \{x, y\} \subseteq \big(x_0 - \delta(\tfrac{\epsilon}{2}, x_0), x_0 + \delta(\tfrac{\epsilon}{2}, x_0)\big) && \text{(Lebesgue's number)} \\
&\Rightarrow \exists x_0 \in [a, b],\ \ |x - x_0| < \delta(\tfrac{\epsilon}{2}, x_0) \text{ and } |y - x_0| < \delta(\tfrac{\epsilon}{2}, x_0) \\
&\Rightarrow \exists x_0 \in [a, b],\ \ |f(x) - f(x_0)| < \tfrac{\epsilon}{2} \text{ and } |f(y) - f(x_0)| < \tfrac{\epsilon}{2}. && \text{(Eq. (7.18))} \\
&\Rightarrow |f(x) - f(y)| < \epsilon. && \text{(triangle inequality)}
\end{aligned}$$

This shows that f is uniformly continuous on $[a, b]$. □

Exercise 7.9. Show that if f is uniformly continuous on (a, b), it can be extended to a continuous function on $[a, b]$. Furthermore, show that such a continuous extension is unique, and is uniformly continuous on $[a, b]$.