Assignment 1: Due 12 October 7pm

Instructions: Submit your hand-written or typed-and-printed solutions to the course assistant during the exercise class on Monday 12 October. Late submissions will not be accepted.

All statements proved in the lectures may be used. Results from exercises in Lectures 1–6 can also be used without proof. You may use properties (4.7)–(4.11) in the lecture notes (exponentiation for rational exponents) without proof.

1. Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be two convergent sequences of real numbers, such that $x_n \leq y_n$ for all $n \geq 1$. Show that
$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

2. Let $p_1, p_2, \ldots$ be a convergent sequence of rational numbers, and $q \in \mathbb{Q}$. Let $x > 0$ be a positive real number. Show that
$$\left(\lim_{n\to\infty} x^{p_n}\right)^q = \lim_{n\to\infty} x^{p_n q}.$$

3. Recall the definition (4.18) of exponentiation for real exponents.

   For positive real numbers $x > 0$ and real exponents $p, q \in \mathbb{R}$, show that

   (a) $1^q = 1$.

   (b) $x^q > 0$.

   (c) $x^{p+q} = x^p x^q$.

   (d) $(x^p)^q = x^{pq}$. (You may need previous questions' results.)

   (e) For $x > 1$,
   $$p < q \;\Leftrightarrow\; x^p < x^q.$$

4. Let $x, \alpha \in \mathbb{R}$ with $x \geq 1$. Show that
$$x^\alpha = \sup\{x^q \,:\, q \in \mathbb{Q},\, q \leq \alpha\}.$$