Assignment 2: Due Monday 7 December 6.40pm

Instructions: Submit your hand-written or typed-and-printed solutions to the course assistant during the exercise class on Monday 7 December. Late submissions will not be accepted. You are not allowed to copy AI-generated solutions. We will record suspicious submissions, and compare with your written exams as evidence.

1. Show that the irrational numbers are dense in $\mathbb{R}$, i.e., given any $x, y \in \mathbb{R}$ with $x < y$, there exists $z \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < z < y$.

2. A rational number x has a unique *lowest terms* representation, $x = \frac{p}{q}, p, q \in \mathbb{Z}$, meaning that

   - $q > 0$;
   - The only positive integer that divides both p and q is 1.

   Define a function $f : \mathbb{R} \to \mathbb{R}$ by

   $$f(x) = \begin{cases} \frac{1}{q}, & x \in \mathbb{Q}, \text{ with } x = \frac{p}{q} \text{ in lowest terms,} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

   Determine the set of points where

   (a) f is continuous;

   (b) f is differentiable.

3. Show that the function

   $$f : x \mapsto \begin{cases} (1+x)^{1/x}, & x \in (-1, 0) \cup (0, \infty), \\ e, & x = 0, \end{cases}$$

is differentiable at 0, and determine the derivative $f'(0)$. You may use the facts

$$\lim_{x\to 0}\frac{e^x-1}{x}=1, \qquad \lim_{x\to 0}\frac{\ln(1+x)}{x}=1, \qquad \lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}=-\frac{1}{2},$$

and other results in the Lecture Notes up until page 115. But *do not use Mean Value Theorems, L'Hôpital's Rule, Taylor approximation, integrals, or their consequences* such as Exercise 8.5.

4. For a sequence $x_1, x_2, \ldots$, its n-th Cesàro mean is $\frac{x_1+x_2+\ldots+x_n}{n}$.

   (a) Consider the sequence $x_n = (-1)^n, n \geq 1$. Show that its corresponding sequence of Cesàro means converges. What is the limit?

   (b) Consider the sequence of partial sums,

   $$s_n = \sum_{k=1}^{n}(-1)^k, \qquad n \geq 1.$$

   Show that its corresponding sequence of Cesàro means converges. What is the limit?