Exercises for Week 11

1. Verify that the function

$$x \mapsto |x| = \begin{cases} x, & x \geq 0, \\ -x, & x \leq 0, \end{cases}$$

is continuous at 0, but not differentiable at 0.

2. Consider the functions

$$f_1 : x \mapsto \begin{cases} x, & x \in \mathbb{Q} \\ -x, & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases} \qquad f_2 : x \mapsto \begin{cases} x^2, & x \in \mathbb{Q} \\ -x^2, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Find their points of differentiability (if any).

3. For negative integer degree, $n \leq -1$, use the definition of derivative to show directly that

$$\frac{d(x^n)}{dx}\Big|_{x_0} = n x_0^{n-1}, \qquad x_0 \in \mathbb{R} \setminus \{0\}.$$

4. Let f, g be real-valued functions which are differentiable at x_0. Show that

   - The *Leibniz rule*

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0). \tag{0.1}$$

     holds.

   - Suppose $g(x_0) \neq 0$. Use the Leibniz rule to deduce that

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{\big(g(x_0)\big)^2}.$$

     Note: g is differentiable at x_0, thus continuous at x_0. Why does $g(x_0) \neq 0$ ensure that $\frac{f}{g}$ is defined on some neighbourhood of x_0? (See Exercise 7.1 of lecture notes).

5. Let f be an injective real-valued function on some neighbourhood W of x_0, which is differentiable at x_0 with $f'(x_0) \neq 0$. Assume that $f^{-1} : f(W) \to \mathbb{R}$ is continuous at $y_0 = f(x_0)$.

   - Show that y_0 is a limit point of $f(W)$, and that

   $$\lim_{y \to y_0 \,:\, y \in f(W) \setminus \{y_0\}} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}. \tag{0.2}$$

   - Assume further that f is continuous on W. Show that f^{-1} is defined on some neighbourhood of y_0, differentiable at y_0, and that the formula

   $$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

   holds.

   Remark: Some authors (like Tao) use the terminology "g is differentiable at y_0" without requiring g to be defined on a neighbourhood of y_0. In that convention, the existence of the limit in (0.2) would be called "differentiability of f^{-1} at y_0", and the expected formula for this "derivative" holds (right side of (0.2)).

   In our convention (which is probably more standard), "g is differentiable at y_0" includes the requirement that g is defined on a neighbourhood of y_0. So, in the above question, an extra assumption on f is needed to conclude that "f^{-1} is differentiable at y_0" in our sense.

6. Consider the following disjoint subsets of $\mathbb{R}$,

   $$A = \left\{ \frac{1}{\ln n} \,:\, n \geq 2 \right\}, \qquad B = \{1, 2, 3, \ldots\}.$$

   Define the function

   $$f : \mathbb{R} \to \mathbb{R}$$
   $$x \mapsto \begin{cases} \frac{1}{\ln 2n}, & x = \frac{1}{\ln n}, n \geq 2, \\ \frac{1}{\ln (2n+1)}, & x = n, n \geq 1, \\ x & x \in \mathbb{R} \setminus (A \sqcup B) \end{cases}$$

   Check that f is injective, and differentiable at $x = 0$ with $f'(0) \neq 0$. Verify that f^{-1} is *not* continuous at $f(0)$.

7. Compute the derivative of the function $x \mapsto (\ln x)^{\ln x}, x > 1$, at each $x_0 > 1$.

8. Prove Fermat's stationary point theorem (Lemma 8.3 of lectures).

9. Show that if f has positive left-sided derivative at x_0, then there exists $\delta > 0$ such that $f(x) < f(x_0)$ for all $x \in (x_0 - \delta, x_0)$.

10. Prove the order completeness property of $\mathbb{R}$ from the MVT. (A hint was given in the lecture notes.)

11. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that if $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly monotone increasing on $[a, b]$.

12. Use Corollary 8.6 of the lecture notes to prove the trigonometric addition identities

$$\begin{aligned} \sin(x \pm y) &= \sin(x)\cos(y) \pm \cos(x)\sin(y), \\ \cos(x \pm y) &= \cos(x)\cos(y) \mp \sin(x)\sin(y), \qquad x, y \in \mathbb{R}. \end{aligned}$$

13. Consider the following functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$,

$$f_1 : x \mapsto \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad f_2 : x \mapsto \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

Determine whether f_1, f_2 are continuous and/or differentiable at $x = 0$.