Exercises for Week 12

1. Suppose $x \equiv x(t)$ is differentiable on an open interval I, with $x'(t) \neq 0$ for all $t \in I$. Deduce from Darboux's Theorem that x is strictly monotone on I. Then use other results proved in earlier lectures to show that $x(I)$ is an open interval, and that the inverse function $x^{-1} : x(I) \to I \subseteq \mathbb{R}$ is differentiable on $x(I)$.

2. Let $a > 0$ and $f : (a, +\infty) \to \mathbb{R}$. Define

$$\begin{aligned} \tilde{f} : (0, a^{-1}) &\to \mathbb{R} \\ t &\mapsto f(t^{-1}). \end{aligned}$$

   Show that the "left-sided limit" $\lim\limits_{x \to +\infty} f(x)$ exists in $\{-\infty\} \sqcup \mathbb{R} \sqcup \{+\infty\}$ iff the right-sided limit $\lim\limits_{t \to 0^+} \tilde{f}(t)$ exists in $\{-\infty\} \sqcup \mathbb{R} \sqcup \{+\infty\}$, in which case,

$$\lim_{x \to +\infty} f(x) = \lim_{t \to 0^+} \tilde{f}(t).$$

   Write down the analogous version of this Exercise for $g : (-\infty, -a) \to \mathbb{R}$ and $\lim\limits_{t \to 0^-} \tilde{g}(t)$.

3. Verify that the definitions of "f tends to $\pm\infty$ at x_0" and "L is the limit of f at $\pm\infty$ have equivalent sequential definitions as stated in the lectures.

4. Prove the possibly-unbounded version of Monotone Convergence Theorem for functions: If f is monotone on $(a, b]$, then

$$\lim_{x \to a^+} f(x) = \begin{cases} \inf\limits_{x \in (a,b)} f(x), & f \text{ bounded, increasing,} \\ \sup\limits_{x \in (a,b)} f(x), & f \text{ bounded, decreasing,} \\ +\infty, & f \text{ unbounded, decreasing,} \\ -\infty, & f \text{ unbounded, increasing.} \end{cases}$$

(See Exercise 7.5 for the first two cases.)

Deduce that if f is monotone on an open interval (a, b), then

$$\lim_{x\to a^+} |f(x)| = +\infty \;\Rightarrow\; \lim_{x\to a^+} f(x) \in \{-\infty, +\infty\}.$$

5. Let $n \geq 1$ and $x_0 \in \mathbb{R}$. Is it logical to compute the following limit using (iterated) L'Hôpital's Rule,

$$\lim_{h\to 0} \frac{(x_0+h)^n - x_0^n}{h} = \cdots = nx_0^{n-1}?$$

6. Let $\tilde{f}$ be continuously differentiable on a neighbourhood of a. Define $f(x) = \tilde{f}(x) - \tilde{f}(a)$ and $g(x) = x - a$. Check that f, g satisfy the assumptions needed in the (two-sided) L'Hôpital's Rule.

7. Use $\lim\limits_{x\to 0} \frac{\ln(1+x)-x}{x^2} = -\frac{1}{2}$ to show that the function

$$f : x \mapsto \begin{cases} (1+x)^{1/x}, & x \in \mathbb{R} \setminus \{0\}, \\ e, & x = 0, \end{cases}$$

is differentiable at $x = 0$. Use Corollary 10.1 to further deduce that it is continuously differentiable at $x = 0$.

Analyze the related limit

$$\lim_{x\to+\infty} x\Big(\big(1+\frac{1}{x}\big)^x - e\Big).$$

8. Analyze the following limits

$$\lim_{x\to 0} \frac{e^x - 1}{\sin x}, \qquad \lim_{x\to 1} x^{\frac{1}{1-x}}, \qquad \lim_{x\to a} \frac{a^x - x^a}{x - a}, \qquad \lim_{x\to 1} \frac{1 - \sin^2(\pi x/2)}{1 - x^2}.$$

Check that L'Hôpital's Rule is applicable in these examples (possibly in several different ways). In each application of L'Hôpital's Rule, discuss whether it simply states a tautology.

9. Let $a \neq b$. What is the result of applying L'Hôpital's Rule to

$$\lim_{x\to 0} \frac{e^{ax} - e^{bx}}{\sin ax - \sin bx}?$$

Alternatively, use the sine addition formula to rewrite this limit in terms of the first case in Question 8.

10. Verify that

$$\lim_{x \to -\infty} e^x = 0.$$

Hence, or otherwise, analyze the one-sided limit

$$\lim_{x \to 0^+} (\tan x)^{1/x^2}.$$