Exercises for Week 13

1. Let f and f' be differentiable at a. Show that

$$\lim_{h\to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2} = (f')'(a).$$

2. Let $b_i, i = 1, \ldots, n$ be positive real numbers. Compute

$$\lim_{x\to 0} \left(\frac{b_1^x + b_2^x + \ldots + b_n^x}{n} \right)^{1/x}.$$

3. In Example 7.2.6 of Lou–Mei–Yang, using L'Hôpital's Rule, two computations of

$$\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x}$$

were given, with different answers. Which computation is wrong, and why?

4. Let f be differentiable on $(a, +\infty)$.

   - Suppose $\lim\limits_{x\to+\infty} f(x)$ exists in $\mathbb{R}$. Explain whether we can apply L'Hôpital's Rule to deduce

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{xf(x)}{x} \stackrel{?}{=} \lim_{x\to+\infty} \frac{f(x)+xf'(x)}{1} = \lim_{x\to+\infty} f(x) + \lim_{x\to+\infty} xf'(x)$$
$$\Rightarrow \lim_{x\to+\infty} xf'(x) = 0.$$

   - Suppose f is a bounded function, and f' is monotone. Show that $\lim\limits_{x\to+\infty} f(x)$ exists in $\mathbb{R}$, and that

$$\lim_{x\to+\infty} xf'(x) = 0. \tag{0.1}$$

5. Suppose $f' = 0$ holds on an interval I. Show that f is a constant function on I. (Use Mean Value Theorem.)

6. Let $x_1, x_2, \ldots$ be a sequence of strictly positive real numbers, and let $\ell \in \{-\infty\} \sqcup \mathbb{R} \sqcup \{+\infty\}$.

   - Show that
   $$\lim_{n\to\infty} x_n = \ell \;\Rightarrow\; \lim_{n\to\infty} (x_1 x_2 \cdots x_n)^{1/n} = \ell.$$
   - Show that
   $$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell \;\Rightarrow\; \lim_{n\to\infty} x_n^{1/n} = \ell.$$

7. Consider the sequences
$$x_n = 2n + n\cos(\ln n), \qquad y_n = n, \qquad n \geq 1.$$
   - By considering suitable subsequences, show that
   $$\underline{\lim_{n\to\infty}} \cos(\ln n) = -1, \qquad \overline{\lim_{n\to\infty}} \cos(\ln n) = +1. \tag{0.2}$$
   - Verify that there are *strict* inequalities
   $$\underline{\lim_{n\to\infty}} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} < \underline{\lim_{n\to\infty}} \frac{x_n}{y_n} < \overline{\lim_{n\to\infty}} \frac{x_n}{y_n} < \overline{\lim_{n\to\infty}} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}.$$
   Hint: MVT for the function $f(x) = x\cos(\ln x)$ may be helpful for working out
   $$\underline{\lim_{n\to\infty}} (x_{n+1} - x_n) \quad \text{and} \quad \overline{\lim_{n\to\infty}} (x_{n+1} - x_n).$$

8. Write out the details of the proof of the liminf-limsup part of the $\frac{0}{0}$ Stolz–Cesàro Theorem.