Exercises for Week 2

1. Discuss with the course assistant if you need clarification on the following logical concepts, which I did not cover in detail in class. It takes some practice to get used to logical writing. A reference is T. Tao's Analysis I, Appendix A.

   Often, we consider a statement of the form $P \Rightarrow Q$, where P, Q are themselves statements. A statement $P \Rightarrow Q$ is sometimes called an "implication".

   When we ask whether the implication "$P \Rightarrow Q$" is true or false, we are asking "(*Assuming* P is true, *then* is Q true?)". We are *not* asking "Is P true/false?" or "Is Q true/false?". If an implication $P \Rightarrow Q$ is true, and the right side is known to be false, then the left side is false as well.

   The *negation* of P is the statement which is true when P is false, and is false when P is true. The negation of P is denoted by $\neg P$.

   The implication $\neg Q \Rightarrow \neg P$ is called the *contrapositive* of $P \Rightarrow Q$. Convince yourself that $\neg Q \Rightarrow \neg P$ and $P \Rightarrow Q$ are equivalent statements. (Either both implications are true, or both implications are false.)

   The implication $Q \Rightarrow P$ is called the *converse* of $P \Rightarrow Q$. Think of some examples to demonstrate that $Q \Rightarrow P$ and $P \Rightarrow Q$ are *not* equivalent statements (i.e., one implication true, the other implication false).

   A statement (Q and $\neg Q$) is always false, no matter what Q itself is. Statements of the form (Q and $\neg Q$) are basic examples of *contradictions*.

   Often, you will encounter "We prove that P is true, by contradiction". In such a proof, one specifies some other statement Q, and argues that the following implication

   $$\neg P \Rightarrow (\underbrace{Q \text{ and } \neg Q}_{\text{contradiction}})$$

   holds. Explicitly, one *assumes* that $\neg P$ is true, then deduces the contradictory statement on the right side. The right side is automatically false, so the left side is false; so P is true.

2. (Optional.) Use Peano axioms to deduce that $S(n) = n + 1, n \in \mathbb{N}$, and that addition on $\mathbb{N}$ is commutative and associative, and has the cancellation property. (Reference: T. Tao's Analysis I, Chapter 2.2)

3. Use induction to prove that every nonzero natural number is the successor of some natural number.

4. In lectures, we saw that Peano axioms for $\mathbb{N}$ allow us to:

   - Justify the validity of the mathematical induction technique.
   - Define $+, \leq$, so the finite subsets $\{k \in \mathbb{N} : 0 \leq k \leq n\}$ make sense.

   Deduce the *strong* version of the mathematical induction principle:

   For each $n \in \mathbb{N}$, let $P(n)$ be a true/false statement. Assume that

   $$\begin{cases} P(0) \text{ true}, \\ \text{for each } n \in \mathbb{N}, \text{ if } P(k) \text{ is true for all } 0 \leq k \leq n, \text{ then } P(S(n)) \text{ is true.} \end{cases}$$

   Then $P(n)$ is true for every $n \in \mathbb{N}$.

5. Use the strong induction principle to deduce the *well-ordering property* of $\mathbb{N}$:

   Every non-empty subset $A \subseteq \mathbb{N}$ has a least element.

   (An element $a \in A$ is called a least element of A if $a \leq a'$ for all $a' \in A$.)

6. (Optional.) Use Peano axioms to derive, for $\mathbb{N}$,

   - Distributive law.
   - The product of two nonzero natural numbers is another nonzero natural number.
   - $\leq$ is a total order.
   - If $a, b \in \mathbb{N}$ and c is a nonzero natural number, then $ac < bc$.
   - Cancellation law for multiplication.

   Continue this, and derive the "obvious" properties of $\leq$ on $\mathbb{Q}$, which we stated in the lectures without proof.

7. Make sure you understand the concepts of *equivalence relation*, *equivalence classes*, and *quotient set* defined by an equivalence relation. In particular, convince yourself that two equivalence classes $[a], [b]$ are disjoint if a is not equivalent to b.

   In particular, check that addition and multiplication are well-defined on $\mathbb{Z}$. (We defined $\mathbb{Z}$ to be the set of equivalence classes of ordered pairs of natural numbers).