Exercises for Week 3

1. Prove Euclid's division Lemma. Then show that for each $x \in \mathbb{Q}$, there exists $q \in \mathbb{Z}$ such that
$$q \leq x < q + 1.$$

2. Let $x, y, z \in \mathbb{Q}$. Prove the following statements,

Non-degeneracy. $|x| = 0$ iff $x = 0$. (Thus $|x - y| = 0$ iff $x = y$.)

Triangle inequality. $|x + z| \leq |x| + |z|$. (Thus $|x - z| \leq |x - y| + |y - z|$.)

- $|x| \leq y$ iff $-y \leq x \leq y$.
- $|xy| = |x||y|$.

3. Show that the rational sequence
$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$$
satisfies the Cauchy property.

4. Show that a rational Cauchy sequence is bounded.

5. Show that the product of two rational Cauchy sequences is another rational Cauchy sequence.

6. Let $b_1, b_2, \ldots$ and $b'_1, b'_2, \ldots$ be equivalent rational Cauchy sequences, and let $a_1, a_2, \ldots$ be another rational Cauchy sequences. Show that the two sequences
$$a_1 b_1, a_2 b_2, \ldots \quad \text{and} \quad a_1 b'_1, a_2 b'_2, \ldots$$
are also equivalent.

7. Let $x_1, x_2, \ldots$ be a rational Cauchy sequence approximating "$\sqrt{2}$", as provided by the Proposition in Lecture 3.

- Show that the rational sequences

$$x_1^2, x_2^2, \ldots, x_n^2, \ldots, \quad \text{and} \quad (x_1+1)^2, (x_2+\frac{1}{2})^2, \ldots, (x_n+\frac{1}{n})^2, \ldots,$$

  are Cauchy, and are equivalent to the constant sequence $2, 2, \ldots, 2, \ldots$.
- Deduce that any other rational Cauchy sequence $x_1', x_2', \ldots$ that approximates "$\sqrt{2}$" is equivalent to $x_1, x_2, \ldots$.
- Show that $x_1, x_2, \ldots$ is not equivalent to any constant rational Cauchy sequence.

8. In Lecture 4, we saw that a nonzero $x \in \mathbb{R}$ can be represented by a rational Cauchy sequence $x_1, x_2, \ldots, x_n, \ldots$ with the property that

$$|x_n| \geq c, \qquad \forall n \geq 1,$$

   for some rational $c > 0$. Show that the reciprocal sequence

$$x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}, \ldots$$

   is Cauchy.

9. Prove that for any $x \in \mathbb{R}$, exactly one of the following holds,

$$x \text{ is negative}, \quad x = 0, \quad x \text{ is positive}.$$

   Below, you need the Archimedean property of $\mathbb{R}$, which may only be introduced in Week 3 lectures.

10. Let x be a positive real number. Show that there exists a positive integer N such that $0 < \frac{1}{N} < x$.

    Deduce that if x, y are real numbers with $x < y$, then there exists a rational number p such that $x < p < y$.

11. Verify that if a sequence of real numbers converges to $L \in \mathbb{R}$ and also to $L' \in \mathbb{R}$, then $L = L'$.

12. Show that the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ converges to 0.