Exercises for Week 4

1. Show that the *geometric series* $T_1, T_2, \ldots, T_n, \ldots$, where

$$T_n = \sum_{1 \leq k \leq n} \frac{1}{2^k}, \qquad n \geq 1,$$

   is a Cauchy sequence.

2. Let $x_1 \geq x_2 \geq \ldots \geq x_n \geq \ldots$ be a (monotone decreasing) sequence of real numbers. Suppose $\lim_{n\to\infty} = x$. Show that $x_n \geq x$ for all $n \geq 1$.

3. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be convergent sequences of real numbers, with respective limits $a, b \in \mathbb{R}$, i.e.,

$$\lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} b_n = b.$$

   - Show that

$$\lim_{n\to\infty} (a_n \pm b_n) = a \pm b,$$
$$\lim_{n\to\infty} a_n b_n = ab.$$

   - If $b \neq 0$ and $b_n \neq 0$ for all $n \geq 1$, show that

$$\lim_{n\to\infty} b_n^{-1} = b^{-1}$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

4. Show that if a sequence of real numbers converges to $L \in \mathbb{R}$, then any subsequence also converges to L.

5. Show that a sequence of real numbers converges to $L \in \mathbb{R}$ iff its even and odd subsequences both converge to $L \in \mathbb{R}$.

6. Let $a_1, a_2, \ldots$ and $c_1, c_2, \ldots$ be sequences of real numbers, such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n.$$

   - Show that if $b_1, b_2, \ldots$ is a sequence of real numbers such that

$$a_n \leq b_n \leq c_n, \qquad \forall n \geq 1,$$

     then it is also convergent, with

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

   - Deduce that if $x_1, x_2, \ldots$ is a sequence of real numbers such that

$$\lim_{n\to\infty} |x_n| = 0,$$

     then

$$\lim_{n\to\infty} x_n = 0.$$

7. Using the squeezing principle, show that the sequence whose n-th term is

$$x_n = \frac{\sin n}{n}$$

   converges to 0.

8. Show that the subset

$$A = \{x \in \mathbb{R} \,:\, x \text{ rational and } x^2 < 2\}$$

   does not have a rational number as its least upper bound.

9. Suppose $0 \leq \lambda < 1$. Show that

$$\lim_{n\to\infty} \lambda^n = 0.$$

   (Use Monotone Convergence Theorem, which may only be introduced in Week 4 lectures.)