Exercises for Week 6

1. Suppose $0 \leq \lambda < 1$. Show that

$$\lim_{n\to\infty} \lambda^n = 0.$$

2. Let $x > 0$ be a positive real number. Show that

$$\lim_{n\to\infty} x^{1/n} = 1.$$

3. Let $x, y > 0$ be positive real numbers, and $q \in \mathbb{R}$. Show that $(xy)^q = x^q y^q$.

4. Continue the argument from (4.20) of the lecture notes to deduce that

$$e = \lim_{n\to\infty} (1 + \frac{1}{n})^{n+1}.$$

5. Let $x, y > 0$ be positive real numbers. Suppose $x \neq 1$. Show that there exists a unique real number α such that $x^\alpha = y$.

   One possible approach:

   - Show that $(1 + a)^n > 1 + a^n$ when a is a positive real number and n is a positive integer. (Induction)
   - Find integers M, N such that $x^M < y < x^N$. (Archimedean property)
   - Like in the proof of the completeness Theorem 3.8, construct a monotone increasing rational sequence $\ell_1, \ell_2, \ldots$ and a monotone decreasing rational sequence $u_1, u_2, \ldots$, such that

$$x^{\ell_n} < y < x^{u_n}, \qquad n \geq 1,$$

     with both sequences converging to the same limit $\alpha \in \mathbb{R}$.
   - Argue that $x^\alpha = y$. Then use monotonicity result in Assignment 1 to deduce uniqueness.

6. Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be two bounded sequences of real numbers, such that

$$x_n \leq y_n, \qquad \forall n \geq 1.$$

Show that

$$\underline{\lim}_{n\to\infty} x_n \leq \underline{\lim}_{n\to\infty} y_n, \qquad \overline{\lim}_{n\to\infty} x_n \leq \overline{\lim}_{n\to\infty} y_n.$$

7. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of real numbers. Show that

$$\begin{aligned} \underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n &\leq \underline{\lim}_{n\to\infty} (a_n + b_n) \\ &\leq \underline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n \\ &\leq \overline{\lim}_{n\to\infty} (a_n + b_n) \leq \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n, \end{aligned}$$

where each inequality is valid whenever the lim-inf/lim-sup on each side exist.