Exercises for Week 7

1. Use the limit point principle to prove the Bolzano–Weierstrass Theorem.

2. Use the Bolzano–Weierstrass Theorem to prove the Monotone convergence theorem.

3. The collection
$$\mathscr{F} = \{(n, n+\tfrac{3}{2}) \,:\, n \in \mathbb{Z}\}$$
is an open cover of the *unbounded* set $\mathbb{R}$.

   The collection
$$\mathscr{F} = \Big\{\Big(\frac{1}{n+2}, \frac{1}{n}\Big) \,:\, n \geq 1\Big\}$$
is an open cover of the *open, bounded* interval $(0, 1)$.

   - For these two examples of open covers, are there any finite subcovers?
   - Regard the second $\mathscr{F}$ as an open cover of the *closed, bounded* interval $[\frac{1}{3}, \frac{2}{3}]$. Find a finite subcover of $\mathscr{F}$.

4. Let f be a real-valued function defined on a neighbourhood W of $x_0 \in \mathbb{R}$. Make sure that you understand why
$$\forall \text{ sequences } x_1, x_2, \ldots \text{ in } W \text{ converging to } x_0,$$
$$\lim_{n\to\infty} f(x_n) = f\Big(\lim_{n\to\infty} x_n\Big).$$
   is equivalent to
$$\forall \text{ sequences } x_1, x_2, \ldots \text{ in } W \setminus \{x_0\} \text{ converging to } x_0,$$
$$\lim_{n\to\infty} f(x_n) = f\Big(\lim_{n\to\infty} x_n\Big).$$

5. Show that the function

$$\begin{aligned} f : \mathbb{R} \setminus \{0\} &\to \mathbb{R} \\ x &\mapsto \frac{1}{x}, \end{aligned}$$

does not satisfy $\lim_{x \to 0} f(x) = L$ for any $L \in \mathbb{R}$.

6. Show that the Dirichlet function (defined in lectures) does not have a limit at any point $x_0 \in \mathbb{R}$.

7. Let f, g be real-valued functions defined on some deleted neighbourhood V of a point $x_0 \in \mathbb{R}$. Suppose the limits

$$\lim_{x \to x_0} f(x) = L, \qquad \lim_{x \to x_0} g(x) = M$$

exist.

   (a) Show that for any $\lambda, \mu \in \mathbb{R}$, we have

$$\lim_{x \to x_0} (\lambda f + \mu g)(x) = \lambda L + \mu M.$$

   (b) Show that $\lim_{x \to x_0} fg(x) = LM$.

   (c) If $M \neq 0$, show that

$$\lim_{x \to x_0} \left(\frac{f}{g} \right)(x) = \frac{L}{M}.$$