Exercises for Week 8

1. Let f be a real-valued function defined on some deleted neighbourhood of $x_0 \in \mathbb{R}$.

   Show that f has limit L at x_0 iff f has left-sided limit L at x_0 and right-sided limit L at x_0.

   Deduce that a real-valued function f is continuous at x_0 iff f is left-continuous and right-continuous at x_0.

2. Let f, g be real-valued functions defined on some deleted neighbourhood V of a point $x_0 \in \mathbb{R}$. Suppose the limits

   $$\lim_{x \to x_0} f(x) = L, \qquad \lim_{x \to x_0} g(x) = M$$

   exist.

   (a) Suppose $L > 0$. Show that there exists some $\delta > 0$ such that $f(x) > 0$ holds for all $x \in V$ with $0 < |x - x_0| < \delta$.

   (b) Suppose there exists $\delta > 0$ such that

   $$f(x) \leq g(x), \qquad \forall x \in V \text{ with } 0 < |x - x_0| < \delta.$$

   Show that $L \leq M$.

   (c) Show that there exists $\delta > 0$ such that the set

   $$\{f(x) : 0 < |x - x_0| < \delta\}$$

   is bounded.

3. Let f be a real-valued function, which is *bounded* on some deleted neighbourhood V of x_0. Show that there exists a sequence $x_1, x_2, \ldots$ in V converging to x_0, such that the sequence $f(x_1), f(x_2), \ldots$ converges. (BWT might be helpful here.)

4. Complete the proof of the Characterization of Intervals Proposition, by checking that it holds for unbounded intervals in $\mathbb{R}$.

5. For this question, use the ϵ-δ definition of limit. Use the Nested Interval Theorem to prove the $\Leftarrow$ part of Proposition 6.7 (The Cauchy criterion for function limit).

6. Show that the function

$$\begin{aligned} f : (0, \infty) &\to \mathbb{R} \\ x &\mapsto \sqrt{x}, \end{aligned}$$

is continuous at every $x_0 \in (0, \infty)$, using the ϵ-δ definition of continuity.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be the "zig-zag" function defined by $x \mapsto |x|$ when $x \in [-1, 1]$, and extended to a 2-periodic function, $f(x + 2) = f(x), \forall x \in \mathbb{R}$. Define the function

$$\begin{aligned} g : (0, \infty) &\to \mathbb{R}, \\ x &\mapsto f(\tfrac{1}{x}). \end{aligned}$$

Show that $\lim\limits_{x \to 0^+} g(x)$ does not exist.

8. Use the Intermediate Value Theorem to prove the order completeness property of $\mathbb{R}$.