Exercises for Week 9

1. Prove the converse of the Heine–Borel Theorem[1]: If $E \subseteq \mathbb{R}$ has the property that any open cover of E has a finite subcover, then E is closed and bounded.

2. Let $E \subseteq \mathbb{R}$, and $f, g : E \to \mathbb{R}$ be real-valued functions which are continuous at a point $x_0 \in E$. Show that:

   (a) For any $\lambda, \mu \in \mathbb{R}$, the function $\lambda f + \mu g$ is continuous at x_0.

   (b) The function fg is continuous at x_0.

   (c) If $g(x_0) \neq 0$, then the function $\frac{f}{g}$ is continuous at x_0.

   (Try both the sequential definition and the ϵ-δ definition of continuity.)

3. Show that

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} x, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases} \end{aligned}$$

   is continuous at only a single point.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be the "zig-zag" function defined by $x \mapsto |x|$ when $x \in [-1, 1]$, and extended to a 2-periodic function, $f(x + 2) = f(x), \forall x \in \mathbb{R}$. Define the function

$$\begin{aligned} g : (0, \infty) &\to \mathbb{R}, \\ x &\mapsto f(\tfrac{1}{x}). \end{aligned}$$

   Show that g does not have a (right-sided) limit at 0.

[1] Some authors refer to the $\Leftrightarrow$ statement as the Heine–Borel Theorem for $\mathbb{R}$.

5. Let f be a monotone increasing function defined on an interval I. Show that if c is an interior point of I, then

$$\lim_{x\to c^+} = \inf\{f(x) \,:\, c < x \in I\}, \qquad \lim_{x\to c^-} = \sup\{f(x) \,:\, c > x \in I\}$$

exist.

6. Let $f : I \to \mathbb{R}$ be a continuous function on an interval I. Suppose f is not strictly monotone on I. Show that there exists $a < b < c$ in I such that

$$f(b) \geq \max\{f(a), f(c)\} \qquad \text{or} \qquad f(b) \leq \min\{f(a), f(c)\}.$$

(You could check this claim on a case-by-case basis.)

7. Apply the continuity of $\ln := \log_e(\cdot)$ to show that

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1.$$

Deduce that

(a) For $x \in \mathbb{R}$,

$$\lim_{n\to\infty} n \ln\big(1 + \frac{x}{n}\big) = x,$$

thus

$$e^x = \lim_{n\to\infty} \big(1 + \frac{x}{n}\big)^n.$$

(b)

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

(c) For any $\alpha \in \mathbb{R}$,

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha.$$

8. Let $x \in \mathbb{R}$. Check that the following limit of partial sums exists:

$$\sum_{k=0}^{\infty} \frac{|x|^k}{k!} \equiv \lim_{m\to\infty} \sum_{k=0}^{m} \frac{|x|^k}{k!}.$$

Use (for example) the Cauchy criterion to deduce that

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \equiv \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x^k}{k!} \quad \text{exists.}$$

9. Show that if a real-valued function f is uniformly continuous on $E \subseteq \mathrm{Dom}(f)$, then it maps Cauchy sequences in E to Cauchy sequences in $\mathbb{R}$.