9 Orientations

Mod-2 counting is "unoriented counting" in the sense that + and − are deemed to be indistinguishable. If we want to count things with integers, then we need an "orientation" specifying which things should count as + and which as −.

9.1 Signed intersections?

In particular, we would like to count intersections "with signs". This can be motivated from our previous study of the mod-2 degree of a map $f : S^1 \to S^1$.

Recall from Prop. 8.3 that f has a lift $\tilde{f} : \mathbb{R} \to \mathbb{R}$, and any such lift has the periodicity property

$$\tilde{f}(t+1) - \tilde{f}(t) = \alpha_1, \qquad t \in \mathbb{R}$$

for some integer α_1. Furthermore, f is homotopic to the standardized map

$$f_{\alpha_1} : \exp(t) \mapsto \exp(\alpha_1 t).$$

Let us first study f_{α_1}. Assume $\alpha_1 \neq 0$, and consider the regular value 1, which has preimage being a finite set comprising $|\alpha_1|$ points,

$$f_{\alpha_1}^{-1}(1) = \{\exp(\lambda/|\alpha_1|) \,:\, \lambda = 0, \ldots, |\alpha_1| - 1\},$$

Let us study the lifted function $\tilde{f}_{\alpha_1} : t \mapsto \alpha_1 t$. The graph of $\tilde{f}_{\alpha_1}$ is a straight line connecting $(0,0)$ to $(1, \alpha_1)$. Notice that $\tilde{f}_{\alpha_1}$ takes integer values at the points $t = \lambda/|\alpha_1|$. Furthermore, the derivative at these points,

$$\left.\frac{d\tilde{f}_{\alpha_1}}{dt}\right|_{t=\lambda/|\alpha_1|} = \alpha_1,$$

is either positive or negative depending on the sign of α_1. This sign indicates whether $f_{\alpha_1}(t)$ is runs counter-clockwise or clockwise, as $\exp(t)$ is varied in the counter-clockwise sense.

Accordingly, we attach a ± sign to $\exp(\lambda/|\alpha_1|) \in f_{\alpha_1}^{-1}(1)$. So the *signed* number of points in the set $f_{\alpha_1}^{-1}(1)$ will be the integer

$$\#f_{\alpha_1}^{-1}(1) = \mathrm{sgn}(\alpha_1)|\alpha_1| = \alpha_1 \;\in\; \mathbb{Z}.$$

Let us now return to the function $f : S^1 \to S^1$, which is homotopic to f_{α_1}. Pick a regular value y for f. Pick some $t_0 \in \mathbb{R}$ such that $f(\exp(t_0)) = y$.

Consider the graph of the lift $\tilde{f}$, as t varies in $[t_0, t_0+1]$. This is a smooth curve connecting $(t_0, \tilde{f}(t_0))$ to $(t_0+1, \tilde{f}(t_0)+\alpha_1)$. Along the way, $\tilde{f}$ must attain the intermediate integer jumps

$$\tilde{f}(t_0) + \lambda, \qquad \lambda = 0, \ldots, \alpha_1.$$

But $\tilde{f}$ may not be monotonic, so $\tilde{f}(t_0) + \lambda$ for other integers λ can also be attained.

The values of $t \in [t_0, t_0 + 1)$ for which $\tilde{f}(t)$ equals $(\tilde{f}(t_0) + \text{integer})$, are precisely those t with the property that

$$f(\exp(t)) = \exp(\tilde{f}(t)) = \exp(\tilde{f}(t_0) + \text{integer}) = \exp(\tilde{f}(t_0)) = f(\exp(t_0)) = y,$$

i.e. $\exp(t) \in f^{-1}(y)$. We attach a $\pm$ sign to $\exp(t)$ according to the sign of the derivative of f there. (By the regularity assumption, this derivative will be nonzero.) So we have a *signed* number of points in $f^{-1}(y)$.

Recall that the homotopy $f \sim_h f_{\alpha_1}$ was constructed at the level of the lifts, $\tilde{f} \sim_h \tilde{f}_{\alpha_1}$. The graph of $\tilde{f}$ is then a "deformed" version of the graph of $\tilde{f}_{\alpha_1}$. A little experimentation shows that the signed number of points in $f^{-1}(y)$ stays unchanged under such deformations. Thus, the integer α_1 has the interpretation as the "signed intersection number of f with the point y", or the integer "degree of f".

We shall formulate the notion of "signed intersection number" precisely, and establish the homotopy invariance. Observe that we had to refer to terms like "counter-clockwise" — this is the notion of an "orientation".

9.2 Oriented bases for vector spaces

Let V be an n-dimensional vector space. A pair of *ordered bases* $\beta = \{v_1, \ldots, v_n\}$ and $\beta' = \{v'_1, \ldots, v'_n\}$ can be related by the unique linear isomorphism A mapping each v_i to v'_i. We say that β and β' are *equivalently oriented* if

$$\operatorname{sgn} \det A > 0.$$

Here, we recall that the determinant is independent of the basis being used to express A as a matrix, and it is nonzero iff A is invertible.

It is easy to see that this is an equivalence relation partitioning the set of ordered bases of V into two equivalence classes. An *orientation* on V is a choice of one of these two equivalence classes.

Suppose V is an oriented vector space, meaning that an orientation has been chosen. A basis for V is *positively-oriented* if its equivalence class coincides with the chosen orientation; otherwise it is *negatively-oriented*. We write $\operatorname{sgn}(\beta) = +$ if β is positively-oriented, and $\operatorname{sgn}(\beta) = -$ otherwise. For the zero vector space, an orientation is defined to be a choice between $+1$ and -1.

Note that an orientation is an extra structure on a vector space V. In mathematical language, the set of orientations on a vector space is a "$\mathbb{Z}_2$-torsor", with no canonical reference orientation. What is unambiguous is a *change* of orientation. For example, the equivalence class of a basis is changed by the following operations:

- replacing a basis vector v by $-v$; more generally, replacing v by cv for $c < 0$;
- swapping the positions of any two basis vectors.

Orientation preserving/reversing isomorphisms. Let V, W be oriented vector spaces, and let $A : V \to W$ be a vector space isomorphism. Then A maps equivalently ordered bases of V to equivalently ordered bases of W. In other words, A either *preserves* or *reverses* orientation of bases, according to whether $\operatorname{sgn}\det A = 1$ or -1; here, A is represented as a matrix with respect to positively-oriented bases of V and W.

Exercise 9.1. Let V_1, V_2, V_3 be oriented vector spaces of the same dimension. Let $A : V_1 \to V_2$ and $B : V_2 \to V_3$ be isomorphisms. Show that $B \circ A : V_1 \to V_3$ reverses orientation iff exactly one of A, B reverses orientation.

Transferring orientation via isomorphism. If only V is oriented, then a vector space isomorphism $A : V \to W$ *induces* an orientation on W, by declaring that a basis β for W is positively oriented iff $A^{-1}\beta$ is a positively oriented basis for V.

For example, suppose we are given an isomorphism $\mathbb{R}^n \cong W$. Such an isomorphism maps the standard ordered basis for $\mathbb{R}^n$ to an ordered basis β for W. Then W becomes *oriented* by taking the equivalence class of β as the *positive* orientation.

Product orientation. If V and W are oriented vector spaces, the *product orientation* on $V \times W$ is the equivalence class of the basis $\{v_1, \ldots, v_n, w_1, \ldots, w_m\}$, where $\{v_1, \ldots, v_n\}$ is positively oriented for V and $\{w_1, \ldots, w_m\}$ is positively oriented for W. Here, we regard v_i, w_j as vectors in $V \times W$ via the natural inclusion maps, i.e., $v_i \equiv (v_i, 0)$ and $w_j \equiv (0, w_j)$. It is easy to see that this does not depend on the choice of basis for V and W.

Exercise 9.2. If V, W are oriented vector spaces, then $V \times W$ and $W \times V$ are isomorphic via the map $F : (v, w) \mapsto (w, v)$ which switches arguments. Discuss whether F preserves or reverses the product orientations on $V \times W$ and $W \times V$.

Direct sum orientation. Above, $V \times W$ is sometimes referred to as an *external direct sum.* Often we will encounter a big vector space U, and two subspaces $V, W \subset U$ such that $V + W = U$ and $V \cap W = \{0\}$. Then we say that $U = V \oplus W$ is the (internal) *direct sum* of V and W. If orientations are given on any two of U, V, W, then an orientation is determined on the third one, by the requirement that the product of the orientation on V with the orientation on W coincides with the orientation on U. In this case, we say that $U = V \oplus W$ is the *oriented* direct sum of V and W.

9.3 Clockwise vs counter-clockwise, left vs right?

In 3-dimensional space, we also often talk about "right-hand rule" when labelling coordinate axes x_1, x_2, x_3, and defining cross products of vectors. Similarly, in two-dimensional space, we often talk about "counter-clockwise convention".

But can we *define* "right-hand rule" without being given a literal right hand as a reference? Can we define "counter-clockwise" without first having a clock dial to refer to?

Physical space X is usually assumed to be a 3-manifold, so locally diffeomorphic to the 3-dimensional Euclidean space. Each tangent space T_xX is isomorphic to $\mathbb{R}^3$. If we pick a local parametrization $\phi : U \to V$, then the standard $\mathbf{e}_i$ on $U \subset \mathbf{R}^3$ are pushed forward under $d\phi : TU \to TV$ to become a local coordinate frame $\{e_1, e_2, e_3\}$. Explicitly, there are vector space isomorphisms

$$d\phi_{\phi^{-1}(x)} : \mathbb{R}^3 \to T_xV = T_xX, \qquad x \in V,$$

smoothly varying in x. So at each $x \in V$, we obtain an ordered basis $\{e_1(x), e_2(x), e_3(x)\}$ for T_xX. We have basically transferred the standard orientation on $\mathbf{R}^3$ to the patch V of X, using local coordinates.

What happens if we change coordinates? For example, let U' be the image of U under the inversion map $P : (x_1, x_2, x_3) \mapsto (-x_1, -x_2, -x_3)$. Instead of identifying V with U, we could equally have identified V with U', using

$$\psi := \phi \circ P : U' \to V.$$

The derivative $d\psi$ also induces orientations on T_xX. However,

$$d\psi = d\phi \circ dP = d\phi \circ P,$$

and P reverses orientation since $\det P = -1$. So ψ and ϕ result in *different* orientations on the T_xX.

Another example: Consider the horizontal plane H in $\mathbf{R}^3$. You may choose to look at H "from above" or from "below". The meaning of "counter-clockwise" depends on such a choice. So a (tangent subspace of a) submanifold may not inherit a canonical orientation from the ambient manifold.

We see that orientations on (tangent spaces of) manifolds are *extra data*. Furthermore, it is not clear whether the local orientations induced by local coordinates can be globalized over the whole manifold.

Ozma problem. In physics, a *parity transformation* P refers to a change of orientation. For a long time, it was believed that fundamental physics laws must be invariant under parity transformations. This means that there is no experimental measurement which can absolutely tell "left" from "right".

In biological systems on Earth, certain biochemistry structures like sucrose, proteins and DNA do have a preferred orientation. But these structures should be viewed as "carrying" the data of a choice of orientation, made at some point in the distant past, along a path (parametrized by time, say).

However, suppose you want to communicate "left" versus "right" to an alien civilization *remotely* (i.e. not in-person). This is known as the *Ozma Problem.* Because the alien biology has no path-connection with the Earth's biology, we do now know how alien DNA twists.

As it turns out, there *is* an experiment which can distinguish left from right! This groundbreaking work was carried out by physicist Chien-Shiung Wu, who discovered in 1956 that the decay of Cobalt-60 nuclei violated invariance under parity transformations P. Somehow, she did not receive the Nobel Prize for this amazing work. The 1957 Nobel prize was awarded to her colleagues Chen-Ning Yang and Tsung-Dao Lee, "for their penetrating investigation of the so-called parity laws which has led to important discoveries regarding the elementary particles".

There is still a catch. In quantum mechanics and particle physics, there is another operation of *charge-conjugation*, called C, which exchanges "matter" with "antimatter". The labels "matter" and "antimatter" are conventions, but what is absolute is that they annihilate each other. By convention, the "stuff" which is leftover on earth is called "matter". Although P is not a fundamental symmetry of physical law, the combination CP was conjectured to remain a symmetry. If the alien civilization was made of "antimatter", the Wu experiment would have a different preference for left/right compared to the human one using matter. So we must first send some physical matter (e.g. arrange an in-person meeting) to the alien, so they can determine whether they are made of earth-matter or earth-antimatter.

In 1964, CP-symmetry was found to be violated as well! Nowadays it is the combination CPT, where T is time-reversal, which is believed to be a fundamental symmetry of physical law.

There is another subtlely in the above discussion, namely the assumption of global *orientability* of space. In other words, it is tacitly assumed that it is possible to consistently assign an orientation everywhere in space, so that the only ambiguity is in which global orientation to use.

For points on the surface of the earth, there is a globally unambiguous outward direction, which we usually call "up". So pointwise, choosing an orientation reduces to choosing an orientation for the horizontal tangent plane — the two senses of rotating about a vertical axis. This is what "turn left" or "turn right" is supposed to distinguish.

Consider a person on the Northern hemisphere speaking Language A, trying to explain to a person on the Southern hemisphere speaking Language B, that they should shake their "right" hands when they meet. Fortunately, one can "propagate" a local notion of "right" consistently across the entire surface of the Earth, because the surface of the Earth is orientable. So they just need to arrange for an in-person cultural exchange, and thereafter, there can be agreement on the "right" convention.

If, however, these people were 2D inhabitants of a Möbius band, then the *path* taken matters…!

9.4 Orientations on manifolds

We saw that with a local parametrization $\phi : U \to V \subset X$, we can orient the tangent spaces $T_xX, x \in V$, using the isomorphisms $d\phi_{\phi^{-1}(x)}$.

Over the entire manifold X, we would like to assign pointwise orientations to all the T_xX, $x \in X$. This should be done in a smooth manner, in the following sense.

Definition 28. A manifold-with-boundary X is *orientable*, if there exists an assignment of orientations to every T_xX, such that each $x \in X$ has a local parametrization $\phi : U \to V$ with the property that $d\phi_u : \mathbb{R}^n \to T_{\phi(u)}X$ is an orientation preserving linear isomorphism for every $u \in U$. A choice of such an assignment is called an *orientation* of X.

There exist *unorientable* manifolds: Möbius strip, Klein bottle, real projective plane, etc.

An oriented manifold-with boundary X can be given the *opposite orientation* by reversing the orientation assigned to every T_xX. The local parametrizations just need to be composed with an orientation-reversing diffeomorphism of Euclidean space (e.g. reversing the first coordinate), to show that the opposite orientation satisfies Definition 28.

When X is equipped with the opposite orientation, it is denoted $-X$.

Proposition 9.1. *Let X be a connected orientable manifold-with-boundary. Then it admits exactly two orientations.*

Proof. Pick a reference orientation on X. Now consider a second orientation, and let S be the subset of X on which the second orientation differs from the reference orientation. This means that for $x \in S$, there are two local parametrizations $\phi_i : U_i \to V_i$ around x, such that $d(\phi_1)_{\phi_1^{-1}(x)}$ preserves orientation for all $\phi_1^{-1}(x) \in U_1$ whereas $d(\phi_2)_{\phi_2^{-1}(x)}$ reverses orientation for $\phi_2^{-1}(x) \in U_2$. Thus, for $x \in V_1 \cap V_2$, the map $d(\phi_1^{-1} \circ \phi_2)_{\phi_2^{-1}(x)} : \mathbb{R}^n \to \mathbb{R}^n$ reverses orientation, and its determinant is negative. But the Jacobian matrix elements of $d(\phi_1^{-1} \circ \phi_2)_u$ depends smoothly on $u \in U_2$, and so does the determinant. Thus $\det(d(\phi_1^{-1} \circ \phi_2)_u) < 0$ holds on some open sub-neighbourhood W of $\phi_2^{-1}(x)$. So the two orientations disagree on the open neighbourhood $\phi_2(W)$ of x. This shows that S is open.

Now let $x \in X \setminus S$. Then $d(\phi_1^{-1} \circ \phi_2)_{\phi_2^{-1}(x)}$ preserves orientation, and has positive determinant. The same argument shows that $X \setminus S$ is open, i.e. S is closed. So S is a clopen subset of the connected space X, thus either $S = \emptyset$ (second orientation is the same as reference orientation) or $S = X$ (second orientation is opposite to the reference orientation).

$\square$

Propagating orientations along paths. (Sketched in lecture.) Pick an initial point $x_0 \in X$, and let $\gamma : I \to X$ be a path with $\gamma(0) = x_0$. Write $x_1 = \gamma(1)$ for the final point. First, assume that there is a local parametrization $\phi : U \to V$ around x_0 such that $\gamma(I) \subset V$. There are isomorphisms

$$d\phi_{\phi^{-1}(x)} : \mathbb{R}^n \to T_x X, \qquad x \in V.$$

If we start with an initial orientation on the initial tangent space $T_{x_0}X$, then we can use the isomorphism

$$d\phi_{\phi^{-1}(x_1)} \circ d\phi^{-1}_{\phi^{-1}(x_0)} : T_{x_0}X \to T_{x_1}X \tag{9.1}$$

to induce an orientation on the final tangent space $T_{x_1}X$.

Unfortunately, a path joining x_0 to x_1 may not be contained in a single coordinate neighbourhood. Instead, by a compactness argument, we partition I into subintervals $[t_{i-1}, t_i], i = 1, \ldots, m$, such that each sub-path $\gamma([t_{i-1}, t_i])$ is contained in some coordinate neighbourhood V_i. Then we apply the previous procedure along each sub-path, to inductively "propagate" the orientation along the entire path. So an orientation on the $T_{x_1}X \equiv T_{\gamma(t_m)}X$ is obtained by sequentially composing

$$O_i^\gamma : d(\phi_i)_{\phi_i^{-1}(\gamma(t_i))} \circ d(\phi_i)^{-1}_{\phi_i^{-1}(\gamma(t_{i-1}))} : T_{\gamma(t_{i-1})}X \to T_{\gamma(t_i)}X.$$

The orientation at x_0 gets transferred to an orientation at $\gamma(t_1)$, which gets transferred to an orientation at $\gamma(t_2)$, and so on, until $\gamma(t_m) = x_1$ is reached.

If we had chosen a different partition $[t_{j-1}, t_j], j = 1, \ldots, m'$ for the path γ, and/or used different coordinate neighbourhoods V'_j, we will still end up with the same orientation at x_1. This claim should be verified, using a similar argument to the proof of Prop. 9.1.

Thus, given any points x_0, x_1, there is a well-defined way to "propagate orientation" along a path γ connecting x_0 to x_1, by applying

$$O^\gamma := O_m^\gamma \circ O_{m-1}^\gamma \circ \ldots \circ O_1^\gamma : T_{x_0}X \to T_{x_1}X.$$

Does the resulting orientation at x_1 depend on the *choice* of path γ?

Let us considering a homotopy of paths $\gamma_s : I \to X$, where the endpoints of each path are fixed to be x_0 and x_1. Fix $s_0 \in I$, and take a partition of γ_{s_0}; thus for $i = 1, \dots, m$, there are compact sets $\gamma_{s_0}([t_{i-1}, t_i])$ contained in connected coordinate neighbourhoods V_i. Some neighbourhood of $\gamma_{s_0}([t_{i-1}, t_i])$ is still contained in V_i, so by continuity, there is some $\delta > 0$ such that

$$\gamma_s([t_{i-1}, t_i]) \subset V_i, \qquad \forall s \in (s_0 - \delta, s_0 + \delta) \cap I,\ \forall i = 1, \dots, m.$$

In other words, sufficiently nearby paths have subpaths sharing the same coordinate neighbourhoods.

The i-th term in the factorization $O^{\gamma_s} = O_m^{\gamma_s} \circ \dots \circ O_1^{\gamma_s}$ can be further factorized as

$$O_i^{\gamma_s} : T_{\gamma_s(t_{i-1})}X \xrightarrow{A_{s,i}} T_{\gamma_{s_0}(t_{i-1})}X \xrightarrow{O_i^{\gamma_{s_0}}} T_{\gamma_{s_0}(t_i)}X \xrightarrow{B_{s,i}} T_{\gamma_s(t_i)}X, \tag{9.2}$$

where $O_i^{\gamma_s}, O_i^{\gamma_{s_0}}$ and $A_{s,i}, B_{s,i}$ linear maps of the form $d(\phi_i)_{(\cdot)} \circ d(\phi_i)_{(\cdot)}^{-1}$ as in Eq. (9.1). By definition,

$$\begin{aligned} O_i^{\gamma_s} &: T_{\gamma_s(t_{i-1})}X \to T_{\gamma_s(t_i)}X \\ O_i^{\gamma_{s_0}} &: T_{\gamma_{s_0}(t_{i-1})}X \to T_{\gamma_{s_0}(t_i)}X \end{aligned}$$

are orientation preserving. We do not know whether $A_{s,i}, B_{s,i}$ preserves orientation. But, for each i, either both are orientation preserving, or both are orientation reversing, in order to be consistent with the factorization Eq. (9.2).

If we use γ_s, then $T_{x_1}X$ is oriented via the map

$$O^{\gamma_s} = \underbrace{B_{s,m}}_{\text{id}} \circ O_m^{\gamma_{s_0}} \circ A_{s,m} \circ B_{s,m-1} \circ O_{m-1}^{\gamma_{s_0}} \circ A_{s,m-1} \circ \dots \circ B_{s,1} \circ O_1^{\gamma_{s_0}} \circ \underbrace{A_{s,1}}_{=\text{id}}$$

to T_{x_0}. If we use γ_{s_0}, then $T_{x_1}X$ is oriented by applying

$$O^{\gamma_{s_0}} = O_m^{\gamma_{s_0}} \circ \dots \circ O_1^{\gamma_{s_0}}.$$

In the first case, the total number of orientation reversing maps in the extra terms $A_{s,i}, B_{s,i}$ is even. We conclude that O^{γ_s} and $O^{\gamma_{s_0}}$ induce the same orientation on $T_{x_1}X$.

We have shown that the orientation on $T_{x_1}X$ induced by the path γ_s is locally constant in s, thus it is independent of s, since I is connected. So homotopic paths connecting x_0 and x_1 induce the same orientation at x_1.

Recall that a *simply-connected* manifold (with boundary) X is a connected manifold such that every smooth map $S^1 \to X$ is homotopic to a constant map. Informally, all round-trips on X can be shrunk to a point. Our discussion on "propagation of orientation" leads to the following basic criterion:

Theorem 9.2. *Every simply-connected manifold-with-boundary is orientable.*

Proof. Choose any orientation at a reference point x_0 of the manifold X. For any other point x, use a path connecting x to x_0 to induce an orientation $\mathcal{O}_x$ at x. The pointwise assignment $x \mapsto \mathcal{O}_x$ is path-independent by the simply-connected hypothesis on X.

We still need to check that this assignment of orientations is smooth. Take local parametrizations $\phi_\alpha : U_\alpha \to V_\alpha$, $\alpha \in \mathcal{I}$, with each V_α connected, and $\{V_\alpha\}_{\alpha\in\mathcal{I}}$ covering the manifold X. Let $x \in V_\alpha$ be given the pointwise orientation $\mathcal{O}_x$ from its connection to x_0. Starting from $\mathcal{O}_x$, the local parametrization ϕ_α can be used to smoothly induce an orientation $\mathcal{O}^\alpha_{x'}$ at any other $x' \in V_\alpha$. By construction, $\mathcal{O}^\alpha_{x'}$ coincides with the one induced along a(ny) path in V_α connecting x to x'. By the simply-connected assumption on X, that path from x to x' is homotopic to the (smoothened) concatenation of the path from x to x_0 followed by the path from x_0 to x'. Therefore $\mathcal{O}_{x'}$ coincides with $\mathcal{O}^\alpha_{x'}$, showing that the former is indeed smoothly assigned. □

9.4.1 Product and boundary orientation

Definition 29. For a product $X \times Y$ of two manifolds, where one of them is allowed to have a boundary, the *product orientation* is the one where each $T_{(x,y)}(X \times Y) = T_xX \times T_yY$ is given the product orientation.

Definition 30. Let X be an oriented n-manifold-with boundary. The *boundary orientation* on ∂X is defined by assigning to each basis $\{v_1, \dots, v_{n-1}\}$ of $T_x(\partial X), x \in \partial X$, the sign

$$\operatorname{sgn}\{v_1, \dots, v_{n-1}\} := \operatorname{sgn}\{v_0, v_1, \dots, v_{n-1}\},$$

where $v_0 \in T_xX$ is any *outward*-pointing tangent vector (Definition 6).

Example 9.1. Consider a point on the boundary $\partial\mathbb{H}^n$ of a half-space. Then $-\mathbf{e}_n$ is an outward-pointing tangent vector. The basis $\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}\}$ has sign $(-1)^n$, since

$$\operatorname{sgn}\{-\mathbf{e}_n, \mathbf{e}_1, \dots, \mathbf{e}_{n-1}\} = -(-1)^{n-1}\operatorname{sgn}\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}, \mathbf{e}_n\} = (-1)^n.$$