INFINITY CATEGORIES O2025
ASSIGNMENT 1 (DUE FRIDAY 17 OCTOBER, 3PM)

(1) Consider $\mathbb{N}$ as a category with objects being the natural numbers, and

$$\mathrm{Hom}_{\mathbb{N}}(m,n) = \begin{cases} \{m \to n\}, & \text{if } m \leq n, \\ \emptyset, & \text{if } m > n, \end{cases}$$

with composition $(m \to n) \circ (n \to p) = (m \to p)$. Let $F : \mathbb{N} \to \mathbb{N}$ be the assignments

$$\begin{aligned} n &\mapsto 2n, & n &\in \mathrm{Ob}(\mathbb{N}), \\ \big(m \to n\big) &\mapsto \big(2m \to 2n\big), & m &\leq n. \end{aligned}$$

Verify that F is a functor. Find a left adjoint functor to F and a right adjoint functor to F.

(2) Let $\mathcal{C}$ be a small category, and $N(\mathcal{C})$ be its nerve (which is a simplicial set $\Delta^{\mathrm{op}} \to \mathbf{Set}$). For each $[n] \in \mathrm{Ob}(\Delta)$, write $N(\mathcal{C})_n := N(\mathcal{C})([n])$.

Describe, in terms of the objects and morphisms of C,

- $N(\mathcal{C})_0$.
- $N(\mathcal{C})_1$.
- $N(\mathcal{C})_2$.
- Let $a, b : [0] \to [1]$ be the (order-preserving) maps

$$a(0) = 0, \qquad b(0) = 1.$$

Describe the functions $N(\mathcal{C})(a) : N(\mathcal{C})_1 \to N(\mathcal{C})_0$ and $N(\mathcal{C})(b) : N(\mathcal{C})_1 \to N(\mathcal{C})_0$.

- For the unique (order-preserving) map $a : [1] \to [0]$, describe $N(\mathcal{C})(a) : N(\mathcal{C})_0 \to N(\mathcal{C})_1$.
- Let $a : [1] \to [2]$ be the (order-preserving) map

$$a(0) = 0, \qquad a(1) = 2.$$

Describe the function $N(\mathcal{C})(a) : N(\mathcal{C})_2 \to N(\mathcal{C})_1$.

(3) Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be functors. Suppose

$$\alpha : LR \Rightarrow \mathrm{Id}_{\mathcal{D}}, \qquad \beta : \mathrm{Id}_{\mathcal{C}} \Rightarrow RL$$

are natural transformations satisfying the triangle identities.

- For each fixed $c \in \mathrm{Ob}(\mathcal{C})$ and $d \in \mathrm{Ob}(\mathcal{D})$, consider the function

$$\begin{aligned} \overline{(-)} : \mathrm{Hom}_{\mathcal{D}}(Lc, d) &\to \mathrm{Hom}_{\mathcal{C}}(c, Rd) \\ f &\mapsto R(f) \circ \beta_c. \end{aligned} \tag{0.1}$$

Show that these $\overline{(-)}$ are bijections.

- Verify that the bijections $\overline{(-)}$ depend naturally on $(c,d) \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D})$. (Thus (0.1) is an adjunction between L and R.)
- What is the unit and counit for the above adjunction $L \dashv R$?

(4) Show that if L and L' are two left adjoint functors for a functor R, then L and L' are naturally isomorphic.

(5) Let $D : \mathcal{I} \to \mathcal{C}$ be a diagram. For each fixed $c \in \mathrm{Ob}(\mathcal{C})$, the set of cones from c to D is denoted

$$\mathrm{Cones}(c, D).$$

If $f : c \to c'$ is a morphism in $\mathcal{C}$, then we get a map

$$f^* : \mathrm{Cones}(c', D) \to \mathrm{Cones}(c, D)$$

by pre-composition. So there is a functor

$$\mathrm{Cones}(-, D) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}.$$

($\mathrm{Cones}(-, D)$ is a presheaf over $\mathcal{C}$.)

- Show that there are also bijections
$$\mathrm{Cones}(c, D) \cong \varprojlim \mathrm{Hom}_{\mathcal{C}}(c, D(-))$$
depending naturally on $c \in \mathrm{Ob}(\mathcal{C})$. (You may use the formula for limits of diagrams in **Set** given in the lectures.)
- Show that if D has a limit $\varprojlim D$, then there is a natural isomorphism
$$h_{\varprojlim D} \equiv \mathrm{Hom}_{\mathcal{C}}(-, \varprojlim D) \cong \mathrm{Cones}(-, D).$$
Conversely, suppose $\mathrm{Cones}(-, D)$ is naturally isomorphic to h_c for some $c \in \mathrm{Ob}(\mathcal{C})$. Show that c is a limit object for D.
- Suppose $R : \mathcal{C} \to \mathcal{B}$ is a right adjoint functor to $L : \mathcal{B} \to \mathcal{C}$. Show that if D has a limit $\varprojlim D$, then $R(\varprojlim D)$ is a limiting object for the diagram $R \circ D : \mathcal{I} \to \mathcal{B}$.

(6) For the adjunction $|\cdot| \dashv \mathrm{Sing}$, determine the unit and counit.