INFINITY CATEGORIES 2025
ASSIGNMENT 2 (DUE TUESDAY 25 NOVEMBER, 1PM)

(1) Let S be a simplicial set. Each "morphism" $f \in S([1])$ has a source object $c \in S([0])$ and a target object $c' \in S([0])$, given by the formulae

$$c = S(d^1)(f), \qquad c' = S(d^0)(f),$$

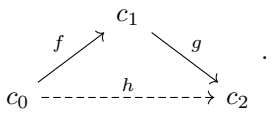
where $d^j : [0] \to [1]$ are the maps skipping j. So we denote f by $c \xrightarrow{f} c'$. Recall that by the Yoneda Lemma, each $f \in S([1])$ corresponds to a unique 1-simplex $\widehat{f} : \Delta^1 \Rightarrow S$ (Remark 3 of lecture notes).

Let $\big(c_0 \xrightarrow{f} c_1\big)$, $\big(c_1 \xrightarrow{g} c_2\big)$ be any two elements of $S([1])$. For $j = 0, 1, 2$, let $s^j : [2] \to [1]$ be the order-preserving map which repeats j. Via the natural isomorphisms $s^0_* : \Delta^{\{1,2\}} \Rightarrow \Delta^1$ and $s^2_* : \Delta^{\{0,1\}} \Rightarrow \Delta^1$, we get

$$\widehat{f} \circ s^2_* : \Delta^{\{0,1\}} \Rightarrow S, \qquad \widehat{g} \circ s^0_* : \Delta^{\{1,2\}} \Rightarrow S.$$

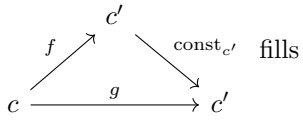
(a) Explain why the above two natural transformations can be combined into a single horn $\Lambda^2_1 \Rightarrow S$.

(b) Suppose $S = \mathcal{C}$ is an ∞-category. Briefly explain why there exists $\sigma \in \mathcal{C}([2])$ whose 0-th face is g and 2-nd face is f,



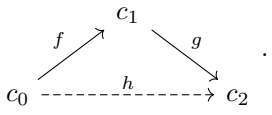
Remark: Suppose $\sigma_0, \ldots, \sigma_{j-1}, \sigma_{j+1}, \ldots, \sigma_n \in S([n-1])$. When we depict these σ_i as the $(n-1)$-dimensional faces of a topological horn $|\Lambda^n_j|$, it is understood that the σ_i have certain coincidence relations amongst their subfaces. The corresponding $(n-1)$-simplices $\widehat{\sigma}_i$, when regarded as $\Delta^{[n]\setminus\{i\}} \Rightarrow S$, $i = 0, \ldots, n, i \neq j$, have overlapping domains $\Delta^{[n]\setminus\{i,k\}}$, and we need to check that they consistently combine into a well-defined horn $\Lambda^n_j \Rightarrow S$. For general $n \geq 2$, This is a rather tedious check for $n \geq 2$, and you may take it for granted.

(2) Let $\mathcal{C}$ be an ∞-category. Recall that morphisms $c \xrightarrow{f} c'$ and $c \xrightarrow{g} c'$ are homotopic if



(a) Use the symmetry property of homotopy, and inner-horn filling property, to argue that

(b) Let $c_1 \xrightarrow{g} c_2$ and $c_0 \xrightarrow{f} c_1$ be morphisms. Let $c_0 \xrightarrow{h} c_2$ be a possible composition of g with f, i.e., there is a 2-simplex $\sigma \in \mathcal{C}([2])$ filling the diagram



Show that $c_0 \xrightarrow{h'} c_2$ is another possible composition of g with f, if and only if h is homotopic to h'.

(c) Let $c_0 \xrightarrow{f'} c_1$, $c_1 \xrightarrow{g'} c_2$, with f homotopic to f' and g homotopic to g'. Show that if h is a possible composition of g and f, and h' is a possible composition of g' and f', then h and h' are homotopic.

(3) Let R be unital ring, and $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, there is a canonical ring isomorphism

$$M_{kn}(R) \cong M_k(M_n(R)),$$

thus a group isomorphism

$$\mathrm{GL}_{kn}(R) \cong \mathrm{GL}_k(M_n(R)).$$

Use these isomorphisms to deduce that $K_1(R) \cong K_1(M_n(R))$.

(4) Let I be an ideal in a unital ring R. The quotient map $R \to R/I$ induces ring homomorphisms $M_n(R) \to M_n(R/I)$ for each $n \in \mathbb{N}$, thus also group homomorphisms $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/I)$. Define

$$\mathrm{GL}_n(R, I) := \ker\big(\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/I)\big).$$

Show that if R' is any other unital ring containing I as an ideal, then $\mathrm{GL}_n(R', I)$ is isomorphic to $\mathrm{GL}_n(R, I)$.

(5) Verify that for a complex inner product space $(V, \langle \cdot | \cdot \rangle)$ and a linear operator $L : V \to V$, the *polarization identity*[1]

$$4\langle x|Ly\rangle = \sum_{k=0}^{3} (-i)^k \langle x + i^k y | L(x + i^k y)\rangle, \qquad x, y \in V,$$

holds.

Let $T \in \mathcal{L}(\mathcal{H})$ be a bounded linear operator on a Hilbert space. Using the above polarization identity, deduce that:

- If T is positive, then T is self-adjoint.
- If T is self-adjoint, then its operator norm satisfies

$$\|T\|_{\mathrm{op}} = \sup_{\|x\| \leq 1} |\langle x|Tx\rangle|.$$

(6) Let T be a trace class operator.

(a) Use polar decomposition to show that $\|T\|_{\mathrm{op}} \leq \|T\|_1$.

(b) Show that $\mathrm{Tr}\, T^* = \overline{\mathrm{Tr}\, T}$.

[1] There are many inter-related identities called "polarization identities".