INFINITY CATEGORIES 2025
ASSIGNMENT 3 (DUE FRIDAY 26 DECEMBER, 3PM)

There are two pages in this assignment!

We sometimes write $\mathrm{GL}(-) = \mathrm{GL}_\infty(-)$ and $\mathrm{E}(-) = \mathrm{E}_\infty(-)$ for convenience, and $R^\times = \mathrm{GL}_1(R)$ denotes the group of units of a unital ring R.

(1) For unital rings R_1, R_2, we write $R_1 \times R_2$ for the cartesian product, with the obvious component-wise product, and unit $(1_{R_1}, 1_{R_2})$. Recall that for non-unital rings I, we have

$$\mathbb{Z} \oplus I, \qquad (n,s)(n',s') = (nn', ns' + n's + ss')$$

being a unital ring with unit $(1,0)$. In lectures, we defined

$$K_0^{\mathrm{nu}}(I) = \ker\big(K_0(\mathbb{Z} \oplus I) \xrightarrow{(\pi_\mathbb{Z})_*} K_0(\mathbb{Z})\big).$$

(Here, we temporarily change notation to K_0^{nu} to distinguish it from K_0 for unital rings.)

(a) For a unital ring R, we write R^{nu} for the **n**on-**u**nital ring obtained by forgetting the unit. Check that

$$\begin{aligned} \mathbb{Z} \oplus R^{\mathrm{nu}} &\to \mathbb{Z} \times R \\ (n, r) &\mapsto (n, r + n1_R) \end{aligned}$$

is an isomorphism of unital rings.

(b) Note that $K_0^{\mathrm{nu}}(R^{\mathrm{nu}})$ makes sense. Explain why

$$K_0^{\mathrm{nu}}(R^{\mathrm{nu}}) \cong K_0(R).$$

(c) If $\phi : I \to J$ is a homomorphism of non-unital rings, define

$$\phi_*^{\mathrm{nu}} : K_0^{\mathrm{nu}}(I) \to K_0^{\mathrm{nu}}(J)$$

to be the restriction of $(\mathrm{id}_\mathbb{Z}, \phi)_* : K_0(\mathbb{Z} \oplus I) \to K_0(\mathbb{Z} \oplus J)$. Verify that $K_0^{\mathrm{nu}}(-)$ is a functor from the category of non-unital rings to the category of abelian groups.

(2) Let $pp' \in M_n(R)$ be idempotents such that $pp' = 0 = p'p$. Show that

$$\begin{pmatrix} p & 0 \\ 0 & p' \end{pmatrix}, \qquad \begin{pmatrix} p+p' & 0 \\ 0 & 0_n \end{pmatrix}$$

are conjugate via some $\mathrm{GL}_{2n}(R)$ matrix. Conclude that

$$[p \oplus (1_n - p)] = [1_n] \qquad \text{in } \mathrm{Idem}(R).$$

(3) In lectures, we defined a projective R-module to be one which can be realized as a direct summand of a free R-module. Show that an R-module P is projective iff whenever $\eta : V \twoheadrightarrow P$ is a surjective R-module homomorphism, there exists an R-module homomorphism $\sigma : P \to V$ such that $\eta \circ \sigma = \mathrm{id}_P$.

(4) Let I be an ideal in R, and suppose

$$0 \to I \to R \xrightarrow{q} R/I \to 0$$

is a *split* short exact sequence, i.e., there is a ring homomorphism $s : R/I \to R$ such that $q \circ s = \mathrm{id}_{R/I}$. *Without using the K_2-to-K_1 connecting map*, show directly that

$$1 \to K_1(R, I) \xrightarrow{j} K_1(R) \xrightarrow{q_*} K_1(R/I) \to 1$$

is a split exact sequence of abelian groups.

(5) [Remark: The purpose of this example is to show that excision fails for K_1.]
Consider the unital rings

$$R = \left\{ \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \,:\, x, y, z \in \mathbb{C} \right\},$$

$$R_0 = \left\{ \begin{pmatrix} x & z \\ 0 & x \end{pmatrix} \,:\, x, z \in \mathbb{C} \right\} \subseteq R,$$

as well as the ideal

$$I = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \,:\, z \in \mathbb{C} \right\}.$$

(a) Compute $K_1(R_0)$. (Hint: modify the arguments in Propositions 7.4–7.6 in the lecture notes.)
(b) By considering question 4, determine the group $K_1(R_0, I)$.
(c) Show that

$$\iota : \mathrm{GL}_1(R, I) \overset{\text{embed}}{\hookrightarrow} \mathrm{GL}(R, I) \overset{\mathrm{mod}\,\mathrm{E}(R,I)}{\longrightarrow} K_1(R, I)$$

is surjective. (Hint: again, consult Propositions 7.4–7.6.)
(d) Show that if we embed $\mathrm{GL}_1(R, I)$ into $\mathrm{GL}_2(R, I)$, then it actually lies within $\mathrm{E}_2(R, I)$. Deduce that $K_1(R, I)$ is the trivial group.