9 Fredholm determinant

9.1 Antisymmetrized tensor products

Let $\mathcal{H}_i, i = 1, 2, \ldots$ be Hilbert spaces, and $x_i \in \mathcal{H}_i$. Consider the map

$$\begin{aligned} x_1 \otimes x_2 : \mathcal{H}_1 \times \mathcal{H}_2 &\to \mathbb{C} \\ (y_1, y_2) &\mapsto \langle y_1|x_1\rangle_{\mathcal{H}_1}\langle y_2|x_2\rangle_{\mathcal{H}_2}. \end{aligned}$$

So $x_1 \otimes x_2$ is a conjugate-bilinear form[17], which we call an *elementary tensor*. The vector space $\mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2$ spanned by such elementary tensors is given the inner product

$$\langle x_1 \otimes x_2 | x_1' \otimes x_2' \rangle := (x_1' \otimes x_2')(x_1, x_2) = \langle x_1|x_1'\rangle\langle x_2|x_2'\rangle, \tag{9.1}$$

and extending sesquilinearly. (We suppress the subscripts in the inner product for cleaner notation.) There are a few things to check:

- Elements of $\mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2$ could be written as a linear combination of elementary tensors in many ways. If such a linear combination μ is the zero conjugate-bilinear form, then we check that for arbitrary $\nu = \sum_{k=1}^N c_k x_{1k} \otimes x_{2k} \in \mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2$,

  $$\langle \nu|\mu\rangle = \sum_{k=1}^N c_n \langle x_{1k} \otimes x_{2k}|\mu\rangle = \sum_{k=1}^N c_n \mu(x_{1k}, x_{2k}) = 0.$$

  So there is no ambiguity in the definition (9.1).

- The inner product (9.1) is positive definite (Exercise).

Then the *tensor product* $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the Hilbert space obtained by completing the inner product space $\mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2$.

Exercise 9.1. If B_i are orthonormal bases for $\mathcal{H}_i, i = 1, 2$, show that $\{e \otimes f\}_{e \in B_1, f \in B_2}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Remark. The algebraic tensor product $\mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2$ is equipped with a bilinear map

$$j : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2, \qquad (x_1, x_2) \mapsto x_1 \otimes x_2.$$

[17] Alternatively, a bilinear form $\mathcal{H}_1^* \times \mathcal{H}_2^* \to \mathbb{C}$, using Riesz representation.

If $L : \mathcal{H}_1 \otimes \mathcal{H}_2 \to V$ is a bilinear map, then

$$\check{L} : \sum_{k=1}^{N} c_k x_{1k} \otimes x_{2k} \mapsto \sum_{k=1}^{N} c_k L(x_{1k}, x_{2k})$$

is the unique linear map $\mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2 \to V$ such that $L = \check{L} \circ j$.

Similarly, there is a (continuous) bilinear map

$$j : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2, \qquad (x_1, x_2) \mapsto x_1 \otimes x_2.$$

However, the following analogous universal property is *false*: For every continuous bilinear map L from $\mathcal{H}_1 \otimes \mathcal{H}_2$ to a Hilbert space $\mathcal{H}$, there exists a unique continuous linear map $\check{L} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}$ such that $L = \check{L} \circ j$. (Exercise: think about the trace in this context.)

We may iterate the tensor product construction, and in particular, we write

$$\mathcal{H}^{\otimes n} = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n \text{ times}}.$$

There is an "antisymmetrizing operator"

$$\begin{aligned} A_n : \mathcal{H}^{\otimes n} &\to \mathcal{H}^{\otimes n} \\ (x_1 \otimes \cdots \otimes x_n) &\to \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} =: \frac{1}{\sqrt{n!}} x_1 \wedge \cdots \wedge x_n. \end{aligned} \tag{9.2}$$

It is not hard to check that A_n is an orthogonal projection, and its range is denoted

$$\Lambda^n \mathcal{H} := A_n \mathcal{H}^{\otimes n}.$$

The normalization on the right side of (9.2) is such that the following identity

holds,

$$\begin{aligned}\langle x_1 \wedge \cdots \wedge x_n | y_1 \wedge \cdots \wedge y_n \rangle &= \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \langle x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} | y_{\tau(1)} \otimes \cdots \otimes y_{\tau(n)} \rangle \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \prod_{i=1}^{n} \langle x_{\sigma(i)} | y_{\tau(i)} \rangle \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \prod_{i=1}^{n} \langle x_i | y_{\tau \circ \sigma^{-1}(i)} \rangle \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{\tau' \in S_n} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau' \circ \sigma) \prod_{i=1}^{n} \langle x_i | y_{\tau'(i)} \rangle \\ &= \sum_{\tau' \in S_n} \operatorname{sgn}(\tau') \prod_{i=1}^{n} \langle x_i | y_{\tau'(i)} \rangle \\ &= \det_n \big(\langle x_i | y_j \rangle \big)_{ij}.\end{aligned} \tag{9.3}$$

In the second equality, we used commutativity in $\mathbb{C}$ to rearrange the order of the factors in each summand. It follows that if $\{e_k\}_{k \in \mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$, so that $\{e_{k_1} \otimes \cdots \otimes e_{k_n}\}_{k_1, \ldots, k_n \in \mathbb{N}}$ is an orthonormal basis for $\mathcal{H}^{\otimes n}$, then the subcollection

$$\{e_{k_1} \wedge \cdots \wedge e_{k_n}\}_{k_1 < \cdots < k_n} \tag{9.4}$$

is an orthonormal basis for $\Lambda^n \mathcal{H}$.

Let $T \in \mathcal{L}(\mathcal{H})$, then $T \otimes \cdots \otimes T \in \mathcal{L}(\mathcal{H}^{\otimes n})$ is defined by

$$(T \otimes \cdots \otimes T)(x_1 \otimes \cdots \otimes x_n) = Tx_1 \otimes \cdots \otimes Tx_n.$$

It may be checked that $\Lambda^n \mathcal{H}$ is an invariant subspace for $T \otimes \cdots \otimes T$, and we denote the restriction to this subspace by

$$\Lambda^n T := (T \otimes \cdots \otimes T)|_{\Lambda^n \mathcal{H}}.$$

Explicitly,

$$\begin{aligned}\Lambda^n T(x_1 \wedge \cdots \wedge x_n) &= \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) Tx_{\sigma(1)} \otimes \cdots \otimes Tx_{\sigma(n)} \\ &= Tx_1 \wedge \cdots \wedge Tx_n.\end{aligned} \tag{9.5}$$

Exercise 9.2. Let $T, T_1, T_2 \in \mathcal{L}(\mathcal{H})$. Show that

$$\begin{aligned} (\Lambda^n T)^* &= \Lambda^n(T^*) \\ |\Lambda^n T| &= \Lambda^n |T| \\ \Lambda^n(T_1 T_2) &= (\Lambda^n T_1)(\Lambda^n T_2). \end{aligned} \tag{9.6}$$

9.2 Determinant for operators close to the identity?

Finite dimensional case. Let F be a linear transformation on a complex vector space V of of finite dimension N. We can construct $V \otimes V$ as the linear span of the elementary tensors, the latter being bilinear maps

$$\begin{aligned} x_1 \otimes x_2 : V^* \times V^* &\to \mathbb{C} \\ (\ell_1, \ell_2) &\mapsto \ell_1(x_1)\ell_2(x_2). \end{aligned}$$

Iterate to get $V^{\otimes n}$, and antisymmetrize to get the subspace $\Lambda^n V$. We also have the linear maps $\Lambda^n F$ on $\Lambda^n V$. If $\{e_k\}_{k=1,\dots,N}$ is a basis for V, then $\{e_{k_1} \wedge \cdots \wedge e_{k_n}\}_{k_1<\dots<k_n}$ is a basis for $\Lambda^n V$. One way to see this is to put an inner product on V making $\{e_k\}_{k=1,\dots,N}$ orthonormal, and note that the Hilbert space constructions of Section 9.1 yield the vector spaces $\Lambda^n V$ after forgetting inner products.

In particular, let $\{e_k\}_{k=1,\dots,N}$ be a basis of generalized eigenvectors exhibiting F in its Jordan canonical form, and write $\lambda_1(F), \dots, \lambda_N(F)$ for the corresponding eigenvalues (repeated according to their multiplicities). Then formula (9.5) says that $e_{k_1} \wedge \cdots \wedge e_{k_n}$ is a generalized eigenvector for $\Lambda^n F$ for the eigenvalue $\lambda_{k_1}(F) \cdots \lambda_{k_n}(F)$. Summing up the latter gives the matrix trace,

$$\operatorname{Tr}_{\Lambda^n V} \Lambda^n F = \sum_{k_1<\cdots<k_n} \lambda_{k_1}(F) \cdots \lambda_{k_n}(F). \tag{9.7}$$

By convention, $\Lambda^0 V = \mathbb{C}$ and $\Lambda^0 F = 1_{\mathbb{C}}$. Then, further summing (9.7) for $n = 0, 1, \dots, N$ gives the formula

$$\sum_{n=0}^{N} \operatorname{Tr}_{\Lambda^n V} \Lambda^n F = \prod_{k=1}^{N} (1 + \lambda_k(F)) = \det_V(1 + F). \tag{9.8}$$

Certainly, for operators F_1, F_2 on V, the multiplicative identity

$$\det_V\big((1 + F_1)(1 + F_2)\big) = \det_V(1 + F_1)\det_V(1 + F_2) \tag{9.9}$$

holds.

Infinite dimensional case. For trace class operators T on an infinite dimensional Hilbert space $\mathcal{H}$, it is conceivable that there is still a sensible notion of

$$\text{“}\sum_{n=0}^{\infty} \operatorname{Tr} \Lambda^n T = \prod_{k=1}^{\infty}(1+\lambda_k(T)) = \det(1+T)\text{ ”}. \tag{9.10}$$

We could define $\det(1+T)$ as the left sum in (9.10). This approach appears, e.g., in B. Simon's *Trace Ideals and Their Applications*. Convergence of this sum amounts to estimating the trace norms $\|\Lambda^n T\|_1$, which is carried out with the help of spectral theory for compact operators[18] on Hilbert spaces, applied to trace class operators. Or we could define $\det(1+T)$ as the middle product in (9.10), as done in Gohberg–Kreĭn's *Introduction to the theory of Linear Nonselfadjoint Operators*.

We will present an approach to $\det(1+T)$, based on extending the finite dimensional formulae (9.8) "by continuity". It is largely based on Gohberg–Goldberg–Kaashoek's *Classes of Linear Operators, Vol.I*, but in the spirit of this course, we deliberately avoid the use of spectral theory, and proceed "algebraically" as far as possible.

We first show that the multiplicative property (9.9) holds when F_1, F_2 are finite rank operators on a possibly infinite dimensional $\mathcal{H}$. A little more work is needed to formulate and prove this.

Definition 9. For $F \in \operatorname{Fin}(\mathcal{H})$, pick any (not necessarily) direct sum decomposition $\mathcal{H} = V \oplus W$ into closed subspaces, such that

$$\dim V < \infty, \qquad F(V) \subseteq V, \qquad W \subseteq \ker F, \tag{9.11}$$

and set

$$\det(1+F) := \det_V\big((1+F)|_V\big). \tag{9.12}$$

Some comments:

- As in the proof of Prop. 8.10, F can be written as $F(-) = \sum_{k=1}^{N} \langle \varphi_k | - \rangle \psi_k$. So we can choose

  $$V = \operatorname{span}\{\varphi_1, \ldots, \varphi_N, \psi_1, \ldots, \psi_N\}, \qquad W = V^{\perp},$$

  to get a decomposition satisfying (9.11).

[18] A linear operator $T : X \to Y$ on normed spaces is *compact* if $\overline{T(A)}$ is compact whenever $A \subseteq X$ is a bounded subset. It can be shown that trace class operators are compact.

- The condition (9.11) means that with respect to $\mathcal{H} = V \oplus W$,

$$1 + F = \begin{pmatrix} 1_V + F|_V & 0 \\ 0 & 1_W \end{pmatrix}.$$

  According to Definition 9,

$$\det(1+F) = \det_V(1_V + F|_V) = \prod_{k=1}^{\dim V} \big(1 + \lambda_k(F|_V)\big), \tag{9.13}$$

  where $\lambda_k(F|_V)$ are the eigenvalues of $F|_V$ listed with algebraic multiplicities.

- Only the nonzero eigenvalues of $F|_V$ contribute to (9.13). They correspond to generalized eigenspaces of F for nonzero eigenvalues, and these generalized eigenspaces lie within V. The same generalized eigenspaces will appear inside V' for any other $V' \oplus W'$ satisfying Eq. (9.11). So Definition 9 does not depend on the choice of $\mathcal{H} = V \oplus W$ satisfying (9.11).

Let $F \in \mathrm{Fin}(\mathcal{H})$, and $S \in \mathcal{L}(\mathcal{H})^\times$. If $\mathcal{H} = V \oplus W$ satisfies (9.11), then since S is a homeomorphism[19], $\mathcal{H} = S(V) \oplus S(W)$ is another direct sum decomposition into closed subspaces, such that

$$\dim S(V) < \infty, \qquad (SFS^{-1})(S(V)) \subseteq S(F(V)) \subseteq S(V), \qquad S(W) \subseteq \ker SFS^{-1}.$$

So we have the conjugation invariance property,

$$\det(1+SFS^{-1}) = \prod_{k=1}^{\dim S(V)} \big(1+\lambda_k(SFS^{-1}|_{S(V)})\big) = \prod_{k=1}^{\dim V} \big(1+\lambda_k(F|_V)\big) = \det(1+F). \tag{9.14}$$

Let $F_1, F_2 \in \mathrm{Fin}(\mathcal{H})$. We have

$$F_i(-) = \sum_{k=1}^{N_i} \langle \varphi_k^{(i)} | - \rangle \psi_k^{(i)}, \qquad i = 1, 2.$$

Choose

$$V = \mathrm{span}\{\varphi_j^{(1)}, \varphi_k^{(2)}, \psi_j^{(1)}, \psi_k^{(2)}\}_{j=1,\ldots,N_1; k=1,\ldots,N_2}.$$

[19] $S^{-1} \in \mathcal{L}(\mathcal{H})$ by definition.

Then for $i = 1, 2$,

$$\dim V < \infty, \qquad F_i(V) \subseteq V, \qquad F_1F_2(V) \subseteq V, \qquad V^\perp \subseteq \ker F_i.$$

It follows that

$$\det\big((1+F_1)(1+F_2)\big) = \det(1+F_1)\det(1+F_2), \qquad \forall\, F_1, F_2 \in \mathrm{Fin}(\mathcal{H}), \tag{9.15}$$

since both sides can be computed with respect to $\mathcal{H} = V \oplus W$.

To extend (9.12) to $\det(1+T)$ for trace class T, it would be necessary to control it in terms of $\|T\|_1$, and Corollary 9.5 below will achieve this.

9.3 Eigenvalues and singular values of matrices

Recall that the *singular values* for a matrix $F \in M_N(\mathbb{C})$, are the eigenvalues of $|F| = \sqrt{F^*F}$. By convention we will write

$$\lambda_1(F), \ldots, \lambda_N(F)$$

for the eigenvalues of F (with multiplicity) arranged in decreasing order of absolute value, as well as

$$s_1(F) \geq \cdots \geq s_N(F)$$

for the singular values of F arranged in decreasing order. Note that trace norm of F is just

$$\|F\|_1 = \mathrm{Tr}_N\,|F| = \sum_{k=1}^{n} s_k(F). \tag{9.16}$$

Lemma 9.1. *For $F \in M_N(\mathbb{C})$,*

$$s_k(F) = \min_{\dim V = N-k+1} \max_{x \in V\,:\,\|x\|=1} \|Fx\|, \qquad k = 1, \ldots, N.$$

Proof. The min-max principle of Courant–Fischer[20] says that for the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ of a self-adjoint matrix H,

$$\lambda_k(H) = \min_{\dim V = N-k+1} \max_{x \in V\,:\,\|x\|=1} \langle x|Hx\rangle, \qquad k = 1, \ldots, N.$$

[20]This can be found in good linear algebra textbooks.

Applying this to $H = F^*F = |F|^2$, and noting that $\langle x|F^*Fx\rangle = \||F|x\|^2 = \|Fx\|^2$ (recall the derivation of Eq. (8.11)), as well as $s_k(F) \equiv \lambda_k(|F|) = \sqrt{\lambda_k(F^*F)}$, we get the claimed min-max characterization of the singular values. □

Although the singular values $s_k(F)$ depend on the inner product on $\mathbb{C}^N$, the product $\prod_{k=1}^N s_k(F)$ does not. To see this, note that F^*F is a self-adjoint matrix, so we can work in a basis in which $F^*F = \mathrm{diag}(s_1^2, \ldots, s_N^2)$, so its positive square root is $|F| = \mathrm{diag}(s_1, \ldots, s_N)$. Then, using properties of the matrix determinant,

$$\Big(\prod_{k=1}^N s_k(F)\Big)^2 = \big(\det_N |F|\big)^2 = \det_N F^*F = (\det_N F^*)(\det_N F) = |\det_N F|^2,$$

and the right side does not refer to the inner product. In fact, by computing $\det_N F$ when F is in its Jordan normal form, we deduce that

$$\prod_{k=1}^N s_k(F) = \prod_{k=1}^N |\lambda_k(F)|, \tag{9.17}$$

where $\lambda_k(F)$ are the eigenvalues of F (counted with algebraic multiplicity). More generally:

Lemma 9.2. *Let F be an operator on an N-dimensional inner product space, and $\{\lambda_k\}_{k=1}^N$, $\{s_k\}_{k=1}^N$ be, respectively, its set of eigenvalues and singular values. Then*

$$\prod_{k=1}^M |\lambda_k(F)| \le \prod_{k=1}^M s_k(F), \qquad M = 1, \ldots, N. \tag{9.18}$$

Proof. Starting from the basis in which F is in Jordan canonical form, apply Gram–Schmidt to get an orthonormal basis $\{e_1, \ldots, e_{N'}\}$ such that

$$Fe_k = \sum_{j=1}^k F_{jk} e_j, \qquad F_{kk} = \lambda_k(F), \qquad k = 1, \ldots, N.$$

Fix $M = 1, \ldots, N$. We can restrict F to $V_M = \mathrm{span}\{e_1, \ldots, e_M\}$, since the latter is an invariant subspace. Then Eq. (9.17) implies that

$$\prod_{k=1}^M |\lambda_k(F)| = \prod_{k=1}^M \big|\lambda_k\big(F|_{V_M\}}\big)\big| = \prod_{k=1}^M s_k\big(F|_{V_M}\big). \tag{9.19}$$

Now, the non-zero singular values of $F|_{V_M}$ are the same as those of

$$\begin{pmatrix} F|_{V_M} & 0 \\ 0 & 0_{V_M{}^\perp} \end{pmatrix} = \underbrace{\begin{pmatrix} F|_{V_M} & * \\ 0 & 0_{V_M^\perp} \end{pmatrix}}_{=F} \underbrace{\begin{pmatrix} 1_{V_M} & 0 \\ 0 & 0_{V_M^\perp} \end{pmatrix}}_{=:P_M=P_M^*=P_M^2}.$$

(F is block upper-triangular because V_M is an invariant subspace.) Then, for any subspace $V \subseteq \mathbb{C}^N$, we have

$$\begin{aligned} \max_{x\in V\,:\,\|x\|=1} \big\|\mathrm{diag}(F|_{V_M}, 0_{V_M^\perp})x\big\| &= \max_{x\in V\,:\,\|x\|=1} \big\|FP_Mx\big\| \\ &\le \max_{x\in V\,:\,\|x\|=1, P_Mx\neq 0} \big\|F(P_Mx/\|P_Mx\|)\big\| \\ &\le \max_{y\in V\,:\,\|y\|=1} \|Fy\|. \end{aligned}$$

So, by the min-max formula for singular values (Lemma 9.1), we deduce that

$$s_k\big(F|_{V_M}\big) \le s_k(F), \qquad k = 1, \dots, M.$$

Putting this into (9.19) gives (9.18). □

Lemma 9.3. *Suppose $a, b \in \{t \in \mathbb{R}^N \,:\, t_1 \ge \cdots \ge t_N\}$ satisfy*

$$\sum_{k=1}^M a_k \le \sum_{k=1}^M b_k, \qquad M = 1, \dots, N. \tag{9.20}$$

Then for any twice differentiable $f : \mathbb{R} \to \mathbb{R}$ with $f'(t), f''(t) \ge 0$, $\forall t \in \mathbb{R}$,

$$\sum_{k=1}^M f(a_k) \le \sum_{k=1}^M f(b_k), \qquad M = 1, \dots, N,$$

holds as well.

Proof. Fix $M = 1, \dots, N$, and write $\Omega_M = \{t \in \mathbb{R}^M \,:\, t_1 \ge \cdots \ge t_M\}$. Temporarily truncate the vectors a, b to their first M components, so that $a, b \in \Omega_M$.

Our task is to show that $\varphi(a) \le \varphi(b)$ holds for the function

$$\varphi : \mathbb{R}^M \to \mathbb{R}, \qquad t \mapsto \sum_{k=1}^M f(t_k),$$

whenever $f', f'' \geq 0$. We can interpolate between $\varphi(a)$ and $\varphi(b)$ using the function

$$\begin{aligned} \psi : [0,1] &\to \mathbb{R} \\ \lambda &\mapsto \varphi(\lambda b + (1-\lambda)a). \end{aligned}$$

It suffices to show that $\psi'(\lambda) \geq 0$ for all $\lambda \in [0,1]$, i.e., ψ is an increasing function, because the desired result

$$\varphi(a) = \psi(0) \leq \psi(1) = \varphi(b)$$

will follow immediately. The verification that $\psi' \geq 0$ is given below.

- Note that $\partial_{t_k}\varphi\big|_t = f'(t_k)$. Then, since $f' \geq 0$ and $f'' \geq 0$, we have

  $$t \in \Omega_M \Leftrightarrow t_1 \geq \cdots \geq t_M \Rightarrow \partial_{t_1}\varphi\big|_t \geq \partial_{t_2}\varphi\big|_t \geq \cdots \geq \partial_{t_M}\varphi\big|_t \geq 0.$$

  Equivalently,

  $$t \in \Omega_M \Rightarrow K\big(\nabla\varphi|_t\big) \text{ has non-negative entries,}$$

  where K is the $M \times M$ matrix

  $$K = \begin{pmatrix} 1 & -1 & & 0 \\ & 1 & \ddots & \\ & & \ddots & -1 \\ 0 & & & 1 \end{pmatrix}.$$

- Ω_M is a convex set. Since $a, b \in \Omega_M$, all their convex combinations lie in Ω_M. Thus

  $$K\big(\nabla\varphi|_{\lambda b+(1-\lambda)a}\big) \text{ has non-negative entries,} \quad \forall\, \lambda \in [0,1].$$

- The first M conditions (9.20) on a, b can be formulated as

  $$L(b-a) \text{ has non-negative entries,}$$

  where L is the $M \times M$ matrix

  $$L = \begin{pmatrix} 1 & & & 0 \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

- Observe that
$$L^{-1} = K^{\mathrm{t}}. \tag{9.21}$$
By the chain rule,
$$\begin{aligned}\psi'(\lambda) &= \nabla\varphi\big|_{\lambda b+(1-\lambda)a} \cdot (b-a) \\ &= \big(K\big(\nabla\varphi|_{\lambda b+(1-\lambda)a}\big)\big) \cdot \big(L(b-a)\big) \qquad \text{(Eq. (9.21))}.\end{aligned}$$
We had argued that, for all $\lambda \in [0,1]$, the vectors in the above dot product have non-negative entries. Therefore, $\psi'(\lambda)$ is non-negative for all $\lambda \in [0,1]$.

□

Proposition 9.4. *Let F be an operator on an N-dimensional inner product space, and $\{\lambda_k\}_{k=1}^N$, $\{s_k\}_{k=1}^N$ be, respectively, its set of eigenvalues and singular values. Then*
$$\prod_{k=1}^{M} \big(1+|\lambda_k(F)|\big) \le \prod_{k=1}^{M} \big(1+s_k(F)\big), \qquad M=1,\dots,N, \tag{9.22}$$
and
$$\sum_{k_1<\cdots<k_M} |\lambda_{k_1}(F)\cdots\lambda_{k_M}(F)| \le \sum_{k_1<\cdots<k_M} s_{k_1}(F)\cdots s_{k_M}(F), \qquad M=1,\dots,N, \tag{9.23}$$

Proof. Without loss of generality, we may assume that $|\lambda_N(F)|$ (thus every $|\lambda_k(F)|$) is non-zero. Then, by Lemma 9.17, all the $s_k(F) > 0$ as well, and we may set
$$a_k = \log|\lambda_k(F)|, \qquad b_k = \log|s_k(F)|, \qquad k=1,\dots,N.$$
Taking the logarithm of the inequalities (9.18) of Lemma 9.2 shows that the above $a, b \in \Omega_N$ satisfies the assumptions of Lemma 9.3.

- Use the function $f(t) = \log(1+e^t)$ in Lemma 9.3, then exponentiate the conclusion, to arrive at (9.22).

- The proof of Lemma 9.3 works to show that

$$\varphi(a) \le \varphi(b)$$

holds for any differentiable $\varphi : \mathbb{R}^M \to \mathbb{R}$ such that

$$\partial_{t_1}\varphi\big|_t \ge \cdots \ge \partial_{t_M}\varphi\big|_t \ge 0, \qquad \forall\, t \in \Omega_M.$$

In particular, we may we use

$$\varphi(t) = \sum_{k_1<\cdots<k_M} \exp(t_{k_1} + \cdots + t_{k_M}),$$

to conclude that (9.23) holds.

□

Corollary 9.5. *In the setting of Prop. 9.4,*

$$|\det_N(1+F)| \le \exp \|F\|_1, \tag{9.24}$$

$$|\det_N(1+F) - 1| \le \exp \|F\|_1 - 1. \tag{9.25}$$

Proof. First,

$$\begin{aligned}
|\det_N(1+F)| &= \prod_{k=1}^{N}(1+\lambda_k(F)) && \text{(Eq. (9.8))}\\
&\le \prod_{k=1}^{N}(1+|\lambda_k(F)|)\\
&\le \prod_{k=1}^{N}(1+s_k(F)) && \text{(Eq. (9.22))}\\
&\le \prod_{k=1}^{N}\exp(s_k(F)) && (1+\mu \le e^{\mu},\ \text{when } \mu \ge 0)\\
&= \exp\Big(\sum_{k=1}^{N} s_k(F)\Big) = \exp\|F\|_1 && \text{(Eq. (9.16))}
\end{aligned}$$

Second,

$$
\begin{aligned}
|\det_N(1+F)-1| &= \Big|\prod_{k=1}^{N}(1+\lambda_k(F))-1\Big| && \text{(Eq. (9.8))}\\
&\le \sum_{M=1}^{N}\sum_{k_1<\cdots<k_M}\big|\lambda_{k_1}(F)\cdots\lambda_{k_M}(F)\big| \\
&\le \sum_{M=1}^{N}\sum_{k_1<\cdots<k_M} s_{k_1}(F)\cdots s_{k_M}(F) && \text{(Eq. (9.23))}\\
&= \prod_{k=1}^{N}(1+s_k(F))-1 \\
&\le \exp\Big(\sum_{k=1}^{N}s_k(F)\Big)-1 && (1+\mu\le e^{\mu},\ \text{when } \mu\ge 0)\\
&= \exp\|F\|_1-1. && \text{(Eq. (9.16))}
\end{aligned}
$$

□

9.4 Fredholm determinant homomorphism

Now we consider possibly infinite dimensional Hilbert spaces $\mathcal{H}$. We mention that for $F\in\mathrm{Fin}(\mathcal{H})$, its trace norm can be computed as

$$\|F\|_1=\|F|_V\|_1,$$

where V is any finite dimensional subspace satisfying (9.12).

Lemma 9.6. *Let $T\in\mathcal{L}^1(\mathcal{H})$, and suppose $1+T$ is invertible (in $\mathcal{L}(\mathcal{H})$). For any sequence $\{F_n\}_{n\in\mathbb{N}}\subseteq\mathrm{Fin}(\mathcal{H})$ converging to T in trace norm, the limit*

$$\lim_{n\to\infty}\det(1+F_n)$$

exists and is independent of the choice of sequence.

Proof. In the first place, $\det(1+F_n)$ is well-defined for each $n\in\mathbb{N}$, as explained in the discussion around Eq. (9.12).

For all large enough n, we have

$$\|(1+T)-(1+F_n)\|_{\mathrm{op}} \le \|(1+T)-(1+F_n)\|_1$$

being small, in which case $1+F_n$ is invertible as well[21]. We claim that $\det(1+F_n), n\in\mathbb{N}$ is a Cauchy sequence (thus convergent). To see this, we first write

$$1+F_m = (1+F_n)\big(1+(1+F_n)^{-1}(F_m-F_n)\big).$$

Then, for all m,n large,

$$\begin{aligned}
&|\det(1+F_m)-\det(1+F_n)| \\
&\quad = \Big|\det(1+F_n)\Big(\det\big(1+(1+F_n)^{-1}(F_m-F_n)\big)-1\Big)\Big| \qquad \text{(Eq. (9.15))} \\
&\quad \le \big(\exp\|F_n\|_1\big)\big(\exp(\|(1+F_n)^{-1}(F_m-F_n)\|_1)-1\big). \qquad \text{(Corollary 9.5)}
\end{aligned}$$

The sequence of norms $\{\|F_n\|_1\}_{n\in\mathbb{N}}$ is bounded, since the F_n converge to T in trace norm by assumption. Also, $\|(1+F_n)^{-1}\|_{\mathrm{op}}$ converges[22] to $\|(1+T)^{-1}\|_{\mathrm{op}}$, while $\|F_m-F_n\|_1\to 0$ as $m,n\to\infty$. It follows that

$$|\det(1+F_m)-\det(1+F_n)|\to 0 \quad \text{as} \quad m,n\to\infty.$$

Let $F_n', n\in\mathbb{N}$ be another finite rank sequence converging to T in trace norm. We similarly obtain

$$|\det(1+F_n')-\det(1+F_n)| \le \big(\exp\|F_n\|_1\big)\big(\exp(\|(1+F_n)^{-1}(F_n'-F_n)\|_1)-1\big).$$

Certainly, $\|F_n'-F_n\|_1 \le \|F_n'-T\|_1+\|F_n-T\|_1\to 0$ as $n\to\infty$, then, as before,

$$\det(1+F_n')-\det(1+F_n)\to 0 \qquad \text{as } n\to\infty.$$

This shows that $\lim_{n\to\infty}\det(1+F_n)=\lim_{n\to\infty}\det(1+F_n')$. □

Because of the density result, Prop. 8.10, given any trace class T, there exists some sequence of finite rank F_n converging to T in trace norm. So, in view of Lemma 9.6, the following is well-defined.

[21]Standard result: The invertible operators on $\mathcal{H}$ form an open set in $\|\cdot\|_{\mathrm{op}}$ topology, by a Neumann series argument.

[22]Standard result: Inversion is continuous in operator norm, by another Neumann series argument.

Definition 10. Let $\mathcal{H}$ be a Hilbert space. The *determinant class* is the subset of operators

$$\mathcal{D}(\mathcal{H}) := \{1 + T \,:\, T \in \mathcal{L}^1(\mathcal{H})\}.$$

The group of *invertible* determinant class operators is denoted

$$\mathcal{D}(\mathcal{H})^\times := \mathcal{D}(\mathcal{H}) \cap \mathcal{L}(\mathcal{H})^\times.$$

The *(Fredholm) determinant* of $1 + T \in \mathcal{D}(\mathcal{H})^\times$ is

$$\det(1 + T) := \lim_{n\to\infty} \det(1 + F_n), \tag{9.26}$$

where $F_n, n \in \mathbb{N}$ is any sequence of finite rank operators converging to T in trace norm.

Remark. $\mathcal{D}(\mathcal{H})^\times$ is indeed a group because of the ideal property of $\mathcal{L}^1(\mathcal{H})$:

- For $1 + T_i \in \mathcal{D}(\mathcal{H})$, $i = 1, 2$, we have

$$(1 + T_1)(1 + T_2) = 1 + \underbrace{T_1 + T_2 + T_1T_2}_{\in\mathcal{L}^1(\mathcal{H})} \in \mathcal{D}(\mathcal{H}).$$

- If $1 + T \in \mathcal{D}(\mathcal{H})^\times$, then a direct calculation shows that

$$(1 + T)^{-1} = 1 - \underbrace{(1 + T)^{-1}T}_{\in\mathcal{L}^1(\mathcal{H})} \in \mathcal{D}(\mathcal{H})^\times.$$

Theorem 9.7. *The Fredholm determinant map* (9.26) *defines a group homomorphism $\mathcal{D}(\mathcal{H})^\times \to \mathbb{C}^\times$, which is invariant under conjugation (by elements of $\mathcal{L}(\mathcal{H})^\times$.)*

Proof. For $1 + T_i \in \mathcal{D}(\mathcal{H})^\times$, $i = 1, 2$, let $F_n, n \in \mathbb{N}$ and $G_n, n \in \mathbb{N}$ be any sequences of finite rank operators converging in trace norm to T_1, T_2 respectively. Then $F_n + G_n + F_nG_n$ converges to $T_1 + T_2 + T_1T_2$. By Definition 10 and Lemma 9.6,

$$\begin{aligned}
\det\big((1 + T_1)(1 + T_2)\big) &= \lim_{n\to\infty}((1 + F_n)(1 + G_n)) \\
&= \big(\lim_{n\to\infty}(1 + F_n)\big)\big(\lim_{n\to\infty}(1 + G_n)\big) \\
&= \det(1 + T_1)\det(1 + T_2).
\end{aligned}$$

Next, let $S \in \mathcal{L}(\mathcal{H})^\times$ and $1+T \in \mathcal{D}(\mathcal{H})^\times$. Then

$$S(1+T)S^{-1} = 1 + \underbrace{STS^{-1}}_{\in \mathcal{L}^1(\mathcal{H})} \in \mathcal{D}(\mathcal{H})^\times.$$

Let $F_n, n \in \mathbb{N}$ be a sequence of finite rank operators converging in trace norm to T, then SF_nS^{-1} converges in trace norm to STS^{-1} (by submultiplicativity, Prop. 8.8). Using Definition 10 and the finite rank conjugation invariance result (9.14), we deduce that

$$\begin{aligned}
\det(S(1+T)S^{-1}) &= \det(1+STS^{-1}) \\
&= \lim_{n\to\infty} \det(1+SF_nS^{-1}) \\
&= \lim_{n\to\infty} \det(1+F_n) \\
&= \det(1+T).
\end{aligned}$$

□

Remark 7. Actually, $\lim_{n\to\infty}(1+F_n)$ in Lemma 9.6 remains well-defined even when $1+T$ is not invertible, in which case that limit is zero. This means that the Fredholm determinant extends to a multiplicative map $\det : \mathcal{D}(\mathcal{H}) \to \mathbb{C}$, via the same formula (9.26). Alternatively, one can define det on all of $\mathcal{D}(\mathcal{H})$ by the formula

$$\det(1+T) = \sum_{n=0}^{\infty} \operatorname{Tr} \Lambda^n T, \qquad T \in \mathcal{L}^1(\mathcal{H}),$$

as mentioned below Eq. (9.10). However, in either approach, one needs to first develop the spectral theory of compact operators, which is not that trivial.