12 Connecting to K_0

Before attempting to figure out $K_n, n \geq 2$, let us construct the K_0 functor and K_1-to-K_0 connecting map, which historically came before the K_2-to-K_1 map. This will provide more evidence or motivation for seeking the $K_n, n \geq 2$. Furthermore, we will see that the K_0 groups are very concrete, and that the K_1-to-K_0 connecting map is related to *index theory*.

12.1 K_0 and relative K_0 groups

Definition 16. For a unital ring R, the abelian group $K_0(R)$ is the Grothendieck group of the monoid $\mathrm{Idem}(R)$ of $\mathrm{GL}_\infty(R)$-conjugacy classes of idempotents in $M_\infty(R)$.

Let us spell out some details of Definition 16.

- The idempotence condition means $p = p^2 \in M_\infty(R)$, and this is preserved under conjugation by $\mathrm{GL}_\infty(R)$.
- The monoid composition $\oplus$ on $\mathrm{Idem}(R)$ is defined at the level of representative idempotents. Namely, if $[p_1], [p_2]$ are two classes in $\mathrm{Idem}(R)$, with representative idempotent matrices $p_1 \in M_{n_1}(R)$ and $p_2 \in M_{n_2}(R)$ respectively, then
$$[p_1] + [p_2] := \left[\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}\right].$$
For well-definedness:
  - Supplementing the representatives p_1, p_2 by extra zeroes results in
$$\begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim_{\text{permutation conj.}} \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
  - If p_i' is conjugate to p_i, by some $\mathrm{GL}_{N_i}(R)$ matrix, then regarding them as $N_i \times N_i$ matrices, we have
$$\left[\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}\right] = \left[\begin{pmatrix} p_1' & 0 \\ 0 & p_2' \end{pmatrix}\right].$$

– We have $[p_1] + [p_2] = [p_2] + [p_1]$ by regarding p_1, p_2 as matrices of the same size, and conjugating by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Note that if $\phi : R \to R'$ is a unital ring homomorphism, then extending it matrix entrywise, we get a multiplicative map from idempotents in $M_\infty(R)$ to idempotents in $M_\infty(R')$, and from $\mathrm{GL}_\infty(R) \to \mathrm{GL}_\infty(R')$. Thus we get a well-defined monoid homomorphism $\phi_* : \mathrm{Idem}(R) \to \mathrm{Idem}(R')$. Recalling that the Grothendieck group construction is functorial, we conclude that

$$\begin{aligned} K_0(-) : \mathbf{Ring} &\to \mathbf{Ab} \\ R &\mapsto G(\mathrm{Idem}(R)) \\ \big(\phi : R \to R'\big) &\mapsto \big(\phi_* : [p_1] - [p_2] \mapsto [\phi(p_1)] - [\phi(p_2)]\big) \end{aligned}$$

is a covariant functor.

Another basic property concerns a Cartesian product $R \times R'$. In this case, idempotents in $M_\infty(R \times R') \cong M_\infty(R) \times M_\infty(R')$ are of the form (p, p'), and $\mathrm{GL}_\infty(R \times R') \cong \mathrm{GL}(R) \times \mathrm{GL}(R')$, and so

$$K_0(R \times R') \cong K_0(R) \oplus K_0(R').$$

We also use the following complemented property of idempotents: If $p \in M_n(R) \subset M_\infty(R)$ is an idempotent, then $(1_n - p)^2 = 1_n - 2p + p = 1_n - p$ is another idempotent.

Exercise 12.1. Let $p = p^2 \in M_n(R)$. Show that

$$\begin{pmatrix} p & 0 \\ 0 & 1_n - p \end{pmatrix}, \qquad \begin{pmatrix} 1_n & 0 \\ 0 & 0_n \end{pmatrix}$$

are conjugate via some $\mathrm{GL}_{2n}(R)$ matrix.

Proposition 12.1. $K_0(\mathbb{Z}) \cong \mathbb{Z}$.

Proof. Let $p = p^2 \in M_n(\mathbb{Z})$. Note that $p\mathbb{Z}^n$ can be regarded as a finite-generated $\mathbb{Z}$-module (the ring $\mathbb{Z}$ acts on the right), i.e., an abelian group. By the structure theorem from algebra, $p\mathbb{Z}^n \cong \mathbb{Z}^r$ for some r (the *rank*), as there are no finite-order elements.

Next, let p, p' be idempotents in $M_n(\mathbb{Z})$. We claim that $p \oplus 0_n$ and $p' \oplus 0_n$ are conjugate idempotents in $M_{2n}(\mathbb{Z})$ iff $p\mathbb{Z}^n$ and $p'\mathbb{Z}'$ are isomorphic. The $\Rightarrow$ direction is immediate. So suppose

$$\phi : p\mathbb{Z}^n \to p'\mathbb{Z}^n$$

is an isomorphism of $\mathbb{Z}$-modules. Noting that $1_n - p$ and $1_n - p'$ are idempotents as well, we may extend φ and φ^{-1} as follows,

$$\begin{aligned} g &: \mathbb{Z}^n \to \mathbb{Z}^n, \qquad g|_{p\mathbb{Z}^n} \equiv \phi, \qquad g|_{(1-p)A^n} \equiv 0, \\ h &: \mathbb{Z}^n \to \mathbb{Z}^n, \qquad h|_{p'\mathbb{Z}^n} \equiv \phi^{-1}, \qquad h|_{(1-p')A^n} \equiv 0. \end{aligned}$$

By construction,

$$gh = p', \qquad hg = p.$$

It follows that

$$\begin{aligned} \overbrace{\begin{pmatrix} g & 1_n - p' \\ 1_n - p & h \end{pmatrix}}^{=:u} \overbrace{\begin{pmatrix} h & 1_n - p \\ 1_n - p' & g \end{pmatrix}}^{u^{-1}} &= \begin{pmatrix} 1_n & 0 \\ 0 & 1_n \end{pmatrix}, \\ u \begin{pmatrix} p & 0 \\ 0 & 0_n \end{pmatrix} u^{-1} = \begin{pmatrix} g & 1_n - p' \\ 1_n - p & h \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0_n \end{pmatrix} \begin{pmatrix} h & 1_n - p \\ 1_n - p' & g \end{pmatrix} &= \begin{pmatrix} p' & 0 \\ 0 & 0_n \end{pmatrix}, \end{aligned} \tag{12.1}$$

as claimed.

The above argument shows that any idempotent $p \in M_\infty(\mathbb{Z})$ is conjugate to 1_r, where r is the rank of p. Therefore, there is a monoid isomorphism $\mathrm{Idem}(\mathbb{Z}) \cong \mathbb{N}$ via the rank map. Taking Grothendieck groups finishes the proof. □

Remark 9. A similar argument shows that $K_0(R) \cong \mathbb{Z}$ when R is a principal ideal domain.

Lemma 12.2. *Let I be an ideal in a unital ring R. For any $u \in \mathrm{GL}_n(R/I)$, there exists a $w \in \mathrm{GL}_{2n}(R)$ which lifts* $\mathrm{diag}(u, u^{-1})$.

Proof. As in the Whitehead Lemma,

$$\mathrm{diag}(u, u^{-1}) = \begin{pmatrix} 1 & 0 \\ u^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-u^{-1} \\ 0 & 1 \end{pmatrix}$$

Pick any (generally non-invertible) lifts $A, B \in M_n(R)$ of u and u^{-1} respectively. Then set

$$w = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-B \\ 0 & 1 \end{pmatrix}.$$

Then w is a lift of $\mathrm{diag}(u, u^{-1})$, and it is a product of invertibles, thus $w \in \mathrm{GL}_{2n}(R)$. □

Definition 17. Let I be an ideal in a unital ring. The *relative K_0-group* is

$$K_0(R, I) := \ker\big((\pi_R)_* : K_0(R \oplus I) \to K_0(R)\big),$$

where $R \oplus I$ is the augmented ring (11.10) and $\pi_R : R \oplus I \to R$ is the projection map. We also define

$$K_0(I) := \ker\big((\pi_{\mathbb{Z}})_* : K_0(\mathbb{Z} \oplus I) \to K_0(\mathbb{Z})\big),$$

where $\pi_{\mathbb{Z}} : \mathbb{Z} \oplus I \to \mathbb{Z}$ is the projection map.

Proposition 12.3. *Let I be an ideal in a unital ring. There is an exact sequence*

$$K_0(R, I) \xrightarrow{\alpha_*} K_0(R) \xrightarrow{q_*} K_0(R/I),$$

where $\alpha : R \oplus I \to R$ is the addition ring homomorphism (11.11).

Above α_* really means the restriction of $\alpha_* : K_0(R \oplus I) \to K_0(R)$ to the subgroup $K_0(R, I)$.

Proof. [Range $\alpha_* \subseteq \ker q_*$.] Let $[(r_1, s_1)] - [(r_2, s_2)] \in K_0(R, I)$, so

$$0 = (\pi_R)_*\big([(r_1, s_1)] - [(r_2, s_2)]\big) = [r_1] - [r_2].$$

Then

$$\begin{aligned} q_* \circ \alpha_*\big([(r_1, s_1)] - [(r_2, s_2)]\big) &= [q(r_1 + s_1)] - [q(r_2 + s_2)] \\ &= [q(r_1)] - [q(r_2)] \\ &= q_*([r_1] - [r_2]) = q_*(0) = 0. \end{aligned}$$

[$\ker q_* \subseteq$ Range α_*.] Next, suppose $q_*\big([p_1] - [p_2]\big) = 0$. We have

$$\begin{aligned} 0 = q_*([p_1] - [p_2]) &= [q(p_1)] - [q(p_2)] \\ \Leftrightarrow [q(p_1) \oplus \dot{r}] &= [q(p_2) \oplus \dot{r}] && (\text{some } \dot{r} \in M_n(R/I)) \\ \Leftrightarrow [q(p_1 \oplus 1_n)] &= [q(p_2 \oplus 1_n)]. && (\text{Exercise (12.1)}) \end{aligned}$$

So, enlarging matrix size if necessary, there exists some $u \in \mathrm{GL}_N(R/I)$ such that

$$q(p_2 \oplus 1_n) = u\big(q(p_1 \oplus 1_n)\big)u^{-1}.$$

Doubling matrix size,

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} q(p_1 \oplus 1_n) & 0 \\ 0 & 0_N \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} q(p_2 \oplus 1_n) & 0 \\ 0 & 0_N \end{pmatrix}.$$

By Lemma 12.2, $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = q(U)$ for some $U \in \mathrm{GL}_{2N}(R)$. So

$$\underbrace{q\left(U \begin{pmatrix} p_1 \oplus 1_n & 0 \\ 0 & 0_N \end{pmatrix} U^{-1}\right)}_{=:p_1'} = \underbrace{q\begin{pmatrix} p_2 \oplus 1_n & 0 \\ 0 & 0_N \end{pmatrix}}_{p_2'}.$$

Therefore,

$$(p_1', p_2' - p_1') \in M_{2N}(R \oplus I),$$

and it is straightforward to check that it is an idempotent. Furthermore,

$$(\pi_R)_*([(p_1', 0)] - [p_1', p_2' - p_1']) = 0,$$

thus $[(p_1', 0)] - [p_1', p_2' - p_1'] \in K_0(R, I)$. We can then apply α_* to this class, obtaining

$$\alpha_*([(p_1', 0)] - [p_1', p_2' - p_1']) = [p_1'] - [p_2'].$$

By construction, $[p_1'] = [p_1]$ and $[p_2'] = [p_2]$, so the last equality says that $[p_1] - [p_2] \in \operatorname{Range} \alpha_*$. □

12.2 Excision

There are obvious homomorphisms $\iota : \mathbb{Z} \to R$ taking $n \mapsto n \cdot 1_R$, and

$$\begin{aligned} \tilde{\iota} : \mathbb{Z} \oplus I &\to R \oplus I \\ (n, s) &\mapsto (n \cdot 1_R, s), \end{aligned}$$

which are such that the following diagram commutes,

$$\begin{array}{ccc} \mathbb{Z} \oplus I & \xrightarrow{\pi_{\mathbb{Z}}} & \mathbb{Z} \\ \downarrow{\scriptstyle \tilde{\iota}} & & \downarrow{\scriptstyle \iota} \\ R \oplus I & \xrightarrow{\pi_R} & R \end{array}.$$

Applying the K_0 functor, we get a commutative diagram,

$$\begin{array}{ccccc} K_0(I) = \ker(\pi_{\mathbb{Z}})_* & \longrightarrow & K_0(\mathbb{Z} \oplus I) & \xrightarrow{(\pi_{\mathbb{Z}})_*} & K_0(\mathbb{Z}) \\ \vdots\, \tilde{\iota}_* & & \downarrow \tilde{\iota}_* & & \downarrow \iota_* \\ K_0(R, I) = \ker(\pi_{\mathbb{R}})_* & \longrightarrow & K_0(R \oplus I) & \xrightarrow{(\pi_R)_*} & R \end{array} ,$$

showing that $\tilde{\iota}_*$ maps $\ker(\pi_{\mathbb{Z}})_* \equiv K_0(I)$ to $\ker(\pi_R)_* \equiv K_0(R, I)$. We continue to write

$$\tilde{\iota}_* : K_0(I) \to K_0(R, I)$$

for the restricted homomorphism.

Proposition 12.4 (Excision for K_0)**.** *Let I be an ideal in a unital ring R. Then $\tilde{\iota}_* : K_0(I) \to K_0(R, I)$ is an isomorphism.*

Surjectivity of $\tilde{\iota}_*$. A general element of $K_0(R, I) = \ker(\pi_R)_*$ has the form $x = [(r_1, s_1)] - [(r_2, s_2)]$, where the idempotent representatives

$$(r_1, s_1),\ (r_2, s_2) \quad \in M_n(R \oplus I)$$

are such that $[r_1] - [r_2] = 0 \in K_0(R)$. After adding some $1_{k,R}$ as a direct summand to r_1 and r_2, they become $\mathrm{GL}_{n+k}(R)$-conjugate. (Because of Exercise 12.1, we may assume that the extra idempotent added to both sides is just some copies of the identity.) So, for convenience, just write r_i instead of $r_i \oplus 1_{k,R}$, and n instead of $n + k$, and we can assume that

$$r_1 = u r_2 u^{-1} \in M_n(R), \qquad \text{for some } u \in \mathrm{GL}_n(R).$$

Now, in $K_0(R, I) \subseteq K_0(R \oplus I)$, we have

$$\begin{aligned} x = [(r_1, s_1)] - [(r_2, s_2)] &= [(r_1, s_1)] - [(u, 0)(r_2, s_2)(u, 0)^{-1}] \\ &= [(r_1, s_1)] - [(u r_2 u^{-1}, u s_2 u^{-1})] \\ &= [(r_1, s_1)] - [(r_1, u s_2 u^{-1})]. \end{aligned}$$

We can add the idempotent

$$(1_{n,R}, 0_{n,I}) - (r_1, u s_2 u^{-1}) = (1_{n,R} - r_1, -u s_2 u^{-1})$$

as a direct summand to the above representatives without changing the K_0 class. This idempotent is complementary to $(r_1, u s_2 u^{-1})$, so the direct sum

with (r_1, us_2u^{-1}) will be conjugate to $(1_{n,R} \oplus 0_{n,R}, 0_{2n})$ by Exercise 12.1. Similarly, conjugating $(r_1, s_1) \oplus (1_{n,R} - r_1, -us_2u^{-1})$ by a suitable $(v, 0) \in \mathrm{GL}_{2n}(R \oplus I)$ to make its R-component $1_{n,R} \oplus 0_{n,R}$. Then we obtain

$$\begin{aligned} x &= \big[(1_{n,R} \oplus 0_{n,R}, \underbrace{v(s_1 \oplus -us_2u^{-1})v^{-1}}_{=:a \in M_{2n}(I)})\big] - [(1_{n,R} \oplus 0_{n,R}, 0_{2n})] \\ &= \tilde{\iota}_*[(1_{n,\mathbb{Z}} \oplus 0_n, a)] - [(1_{n,\mathbb{Z}} \oplus 0_n)]. \end{aligned} \tag{12.2}$$

[Injectivity of $\tilde{\iota}_*$.] Let x be a general element of $K_0(I) \subseteq K_0(\mathbb{Z} \oplus I)$. It can be written as $x = [(m_1, s_1)] - [(m_2, s_2)]$, where m_1, m_2 are idempotent matrices in some $M_n(\mathbb{Z})$, which must satisfy

$$[m_1] - [m_2] = 0 \in K_0(\mathbb{Z}).$$

After adding the trivial class $[(1_n, 0_n) - (m_2, s_2)] - [(1_n, 0_n) - (m_2, s_2)]$, and conjugating by some suitable $u \in \mathrm{GL}_{2n}(\mathbb{Z} \oplus I)$ matrix (Exercise 12.1), we may instead write

$$\begin{aligned} x &= [(m_1, s_1)] - [(1_n, 0_n)], \qquad (m_1, s_1) \in M_{2n}(\mathbb{Z} \oplus I), \\ [m_1] - [1_n] &= 0 \in K_0(\mathbb{Z}). \end{aligned}$$

By Prop. 12.1, $gm_1g^{-1} = 1_n \oplus 0_n$ for some $g \in \mathrm{GL}_{2n}(\mathbb{Z})$. Then

$$x = [(g^{-1}(1_n \oplus 0_n)g, s_1)] - [(1_n, 0_n)] = [(1_n \oplus 0_n, \underbrace{gs_1g^{-1}}_{=:t \in M_{2n}(I)})] - [(1_n, 0_n)]. \tag{12.3}$$

Now assume $\tilde{\iota}_*(x) = 0$, i.e.,

$$0 = [(1_{n,R} \oplus 0_{n,R}, t)] - [(1_{n,R}, 0_n)] \in K_0(R, I) \subseteq K_0(R \oplus I).$$

This means that after stabilizing (increasing n) if necessary, there is a conjugacy relation

$$(u, a)(1_{n,R} \oplus 0_{n,R}, t)(u, a)^{-1} = (1_{n,R} \oplus 0_{n,R}, 0_{2n}), \qquad (u, a) \in \mathrm{GL}_{2n}(R \oplus I)$$

For the R-component, the above equation is $u(1_{n,R} \oplus 0_{n,R})u^{-1} = 1_{n,R} \oplus 0_{n,R}$. Therefore we also have

$$\begin{aligned} (1_{n,R} \oplus 0_{n,R}, 0_{2n}) &= (u^{-1}, 0)(1_{n,R} \oplus 0_{n,R}, 0_{2n})(u, 0) \\ &= (u^{-1}, 0)(u, a)(1_{n,R} \oplus 0_{n,R}, t)(u, a)^{-1}(u^{-1}, 0)^{-1} \\ &= (1_{2n,R}, u^{-1}a)(1_{n,R} \oplus 0_{n,R}, t)(1_{2n,R}, u^{-1}a)^{-1}. \end{aligned}$$

We can view $(1_{2n,R}, u^{-1}a)$ as an invertible in $\mathrm{GL}_{2n}(\mathbb{Z} \oplus I)$ (embedded in $\mathrm{GL}_{2n}(R \oplus I)$). So, dropping the R in the last equation, we have deduced that

$$(1_n \oplus 0_n, 0_{2n}) = (1_{2n}, u^{-1}a)(1_n \oplus 0_n, t)(1_{2n}, u^{-1}a)^{-1} \quad \text{in } M_{2n}(\mathbb{Z} \oplus I),$$

therefore

$$x = [(1_n \oplus 0_n, t)] - [(1_n, 0_n)] = [(1_n, 0_n)] - [(1_n, 0_n)] = 0 \quad \text{in } K_0(I) \subseteq K_0(\mathbb{Z} \oplus I).$$

□

Proposition 12.4 says that $K_0(R, I)$ actually depends only on I. Namely, we can "excise" most of R away, and use only the minimal unitization $\mathbb{Z} \oplus I$ for the definition of $K_0(I)$.

12.3 K_1-to-K_0 connecting map

We shall define a connecting homomorphism $\partial : K_1(R/I) \to K_0(R, I)$. Because of the excision property, it will suffice to construct $\check{\partial} : K_1(R/I) \to K_0(I)$, and then compose with $\tilde{\iota}_*$ to get ∂. Recall that

$$K_0(I) = \ker((\pi_{\mathbb{Z}})_* : K_0(\mathbb{Z} \oplus I) \to K_0(\mathbb{Z})),$$

so its elements are represented by (formal differences of) idempotent matrices over $\mathbb{Z} \oplus I$. *Instead of writing* $(m, s) = (m, 0) + (0, s) \in \mathbb{Z} \oplus I$, *we will often just write* $m + s$.

Definition 18. Let $[u] \in K_1(R/I)$, with representative invertible $u \in \mathrm{GL}_n(R/I)$. Pick any $w \in \mathrm{GL}_{2n}(R)$ such that $q(w) = \mathrm{diag}(u, u^{-1})$, and write $p_n = \begin{pmatrix} 1_n & 0 \\ 0 & 0_n \end{pmatrix}$. Then define

$$\begin{aligned} \check{\partial} : K_1(R/I) &\to K_0(I) \\ [u] &\mapsto [wp_nw^{-1}] - [p_n]. \end{aligned} \tag{12.4}$$

The formula (12.4) is informed by the example of the classical Fredholm index, as explained in Section 12.5.

Remark. Composing with $\tilde{\iota}_*$, we also have

$$\begin{aligned} \partial := \tilde{\iota}_* \circ \check{\partial} : K_1(R/I) &\to K_0(R, I) \\ [u] &\mapsto \big[(p_n, wp_nw^{-1} - p_n)\big] - [(p_n, 0)], \end{aligned} \tag{12.5}$$

where $p_n = \mathrm{diag}(1_{n,R}, 0_{n,R})$.

Here are the checks that (12.4) makes sense.

- By Lemma 12.2, $\mathrm{diag}(u, u^{-1})$ always admits some invertible lift $w \in \mathrm{GL}_{2n}(R)$.
- On the right side of (12.4), p_n refers to $(p_n, 0_{2n,I}) \in M_{2n}(\mathbb{Z} \oplus I)$, so $[p_n] \in \mathrm{Idem}(\mathbb{Z} \oplus I)$. The meaning of $[wp_nw^{-1}] - [p_n]$ needs a bit more explanation. Particularly, why isn't this just zero?

  A priori, $w \in M_{2n}(R)$, while $p_n \in M_{2n}(\mathbb{Z} \oplus I)$, so wp_nw^{-1} doesn't really make sense. It actually means $wp_{n,R}w^{-1}$, where $p_{n,R} = \mathrm{diag}(1_{n,R}, 0_{n,R})$. In other words, $wp_{n,R}w^{-1}$ is an idempotent in $M_{2n}(R)$. Nevertheless, we have

  $$q\left(wp_{n,R}w^{-1} - p_{n,R}\right) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1_{n,R} & 0 \\ 0 & 0_{n,R} \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} - p_{n,R} = 0,$$

  so $wp_{n,R}w^{-1} - p_{n,R} \in M_{2n}(I)$. This means that

  $$\big(p_n, wp_{n,R}w^{-1} - p_{n,R}\big) \in M_{2n}(\mathbb{Z} \oplus I).$$

  Abusing notation, we simply write

  $$wp_nw^{-1} \equiv \big(p_n, wp_{n,R}w^{-1} - p_{n,R}\big),$$

  and then its $\mathrm{GL}(\mathbb{Z} \oplus I)$-conjugacy equivalence class makes sense,

  $$[wp_nw^{-1}] \in \mathrm{Idem}(\mathbb{Z} \oplus I).$$

  Now

  $$\begin{aligned} (\pi_{\mathbb{Z}})_*\big([wp_nw^{-1}] - [p_n]\big) &\equiv (\pi_{\mathbb{Z}})_*\big([p_n, wp_{n,R}w^{-1} - p_{n,R}] - [(p_n, 0_{2n,I})]\big) \\ &= [p_n] - [p_n] = 0 \in K_0(\mathbb{Z}), \end{aligned}$$

  so $[wp_nw^{-1}] - [p_n] \in K_0(I)$ by Definition 17.

- Suppose $w' \in \mathrm{GL}_{2n}(R)$ is another choice of lift of $\mathrm{diag}(u, u^{-1})$. Clearly

$$(w'w^{-1})wp_{n,R}w^{-1}(w'w^{-1})^{-1} = w'p_{n,R}(w')^{-1}.$$

At first, this is a conjugacy relation in $M_{2n}(R)$. But note that $q(w'w^{-1}) = 1_{2n,R/I}$, therefore we may write

$$w'w^{-1} = 1_{2n,R} + s, \qquad \text{for some } s \in M_{2n}(I),$$

and the above conjugacy relation becomes

$$(1_{2n,R} + s)wp_{n,R}w^{-1}(1_{2n,R} + s)^{-1} = w'p_{n,R}(w')^{-1}.$$

But we had seen that $wp_{n,R}w^{-1} \in M_{2n}(R)$ can be viewed as $wp_nw^{-1} \in M_{2n}(\mathbb{Z} \oplus I)$; likewise for $w'p_{n,R}(w')^{-1}$. So we have the conjugacy relation

$$(1_{2n} + s)wp_nw^{-1}(1_{2n} + s)^{-1} = w'p_n(w')^{-1} \quad \in \ M_{2n}(\mathbb{Z} \oplus I),$$

as shorthand for

$$(1_{2n}, s)(p_n, wp_{n,R}w^{-1} - p_{n,R})(1_{2n}, s)^{-1} = \big(p_n, wp_{n,R}(w')^{-1} - p_{n,R}\big).$$

This shows that

$$[w'p_n(w')^{-1}] = [wp_nw^{-1}] \in \mathrm{Idem}(\mathbb{Z} \oplus I)$$

does not depend on the choice of lift w.

- If u is regarded as a larger matrix (with extra 1s), then we can choose the same w but with some extra 1s. It is not hard to see from the formula (12.5) that the extra 1s just cancel out.

- Suppose $[u'] = [u]$ in $K_1(R/I)$. So $u^{-1}u' \in \mathrm{E}(R/I)$, and also $u(u^{-1}u')^{-1}u^{-1} = u(u')^{-1} \in \mathrm{E}(R/I)$ by normality. So there exist $E, F \in \mathrm{E}(R)$ such that

$$q(E) = u^{-1}u', \qquad q(F) = u(u')^{-1}.$$

Then

$$w' = w \cdot \mathrm{diag}(E, F) \overset{q}{\mapsto} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} u^{-1}u' & 0 \\ 0 & u(u')^{-1} \end{pmatrix} = \begin{pmatrix} u' & 0 \\ 0 & (u')^{-1} \end{pmatrix}.$$

According to (12.4), we would have

$$\begin{aligned}\check{\partial}[u'] &= [w'p_n(w')^{-1}] - [p_n] \\ &= \left[w\,\mathrm{diag}(E,F)\mathrm{diag}(1_n,0_n)\mathrm{diag}(E^{-1},F^{-1})w^{-1}\right] - [p_n] \\ &= \left[w\mathrm{diag}(1_n,0_n)w^{-1}\right] - [p_n] \\ &= \check{\partial}[u].\end{aligned}$$

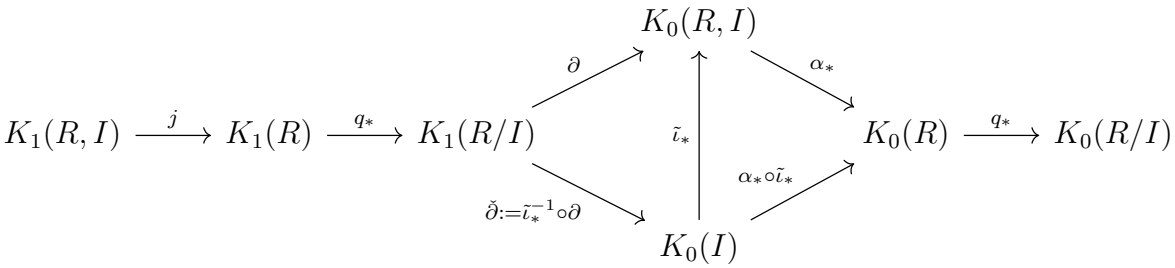
- For the homomorphism property, just note that

$$\check{\partial}([u_1]\cdot[u_2]) = \check{\partial}[u_1 \oplus u_2],$$

then follow the definition (12.4) to deduce the result $\check{\partial}[u_1] + \check{\partial}[u_2]$.

12.4 Exactness of connecting map

Theorem 12.5. *Let I be an ideal in a unital ring. The horizontal rows in the following diagram*



are exact.

Proof. We had already seen that K_1 is half-exact, and that $K_0(R,I) \to K_0(R) \to K_0(R/I)$ is exact. By construction, $K_0 \to K_0(R) \to K_0(R/I)$ inherits the exactness at $K_0(R)$. It remains to check that $\check{\partial}$ makes the lower sequence exact at $K_1(R/I)$ and $K_0(R,I)$. Then ∂ makes the upper sequence exact as well.

- [$\mathrm{Range}\, q_* \subseteq \ker \check{\partial}$.] Straightforward: recalling that $p_n = 1_n \oplus 0_n$, we have, for any $U \in \mathrm{GL}_n(R)$,

$$\begin{aligned}\check{\partial}\circ q_*[U] &= \check{\partial}[q(U)] \\ &= \left[(U\oplus U^{-1})p_n(U^{-1}\oplus U)\right] - [p_n] \\ &= [p_n] - [p_n] = 0.\end{aligned}$$

- [Range $\check{\partial} \subseteq \ker(\alpha \circ \tilde{\iota})_*$.] Let

$$\begin{aligned}\check{\partial}[u] &= [wp_n w^{-1}] - [p_n] \in \operatorname{Range} \partial \\ &\equiv \big[\big(p_n, wp_{n,R}w^{-1} - p_{n,R}\big)\big] - [(p_n, 0_{2n,I})].\end{aligned}$$

Then

$$\begin{aligned}(\alpha \circ \tilde{\iota})_* \circ \partial[u] &= [wp_{n,R}w^{-1}] - [p_{n,R}] \\ &= 0. \qquad\qquad (\text{since } w \in \mathrm{GL}_{2n}(R))\end{aligned}$$

- [Range $q_* \supseteq \ker \check{\partial}$.] Suppose $\check{\partial}[u] = 0$. This means that

$$\big[(p_n, wp_{n,R}w^{-1} - p_{n,R})\big] = [p_n] \text{ in } K_0(I) \subseteq K_0(\mathbb{Z} \oplus I).$$

By increasing n if necessary, we see that $(p_n, wp_{n,R}w^{-1} - p_{n,R})$ is conjugate to $(p_n, 0)$ via some $(v, a) \in \mathrm{GL}_{2n}(\mathbb{Z} \oplus I)$. Here, $v \in \mathrm{GL}_{2n}(\mathbb{Z})$ while $a \in M_{2n}(I)$.

Write (v^{-1}, b) for the inverse of (v, a). Note that this also means $(v + a)^{-1} = (v^{-1} + b)$ (use the α homomorphism). Spelling out the above-mentioned conjugacy relation,

$$\begin{aligned}(p_n, wp_{n,R}w^{-1} - p_{n,R}) &= (v, a)(p_n, 0)(v, a)^{-1} \\ &= (vp_n, ap_n)(v^{-1}, b) \\ &= \big(vp_nv^{-1}, vp_nb + ap_nv^{-1} + ap_nb\big) \\ &= \big(vp_nv^{-1}, (v + a)p_n(v^{-1} + b) - vp_nv^{-1}\big) \\ &= \big(p_n, (v + a)p_n(v + a)^{-1} - p_n\big) \\ &= \big(p_n, (v + a)v^{-1}p_nv(v + a)^{-1} - p_n\big).\end{aligned}$$

To compare the I-components, we embed inside R, Thus,

$$wp_{n,R}w^{-1} = (\iota(v) + a)\iota(v)^{-1}p_{n,R}\iota(v)(\iota(v) + a)^{-1}.$$

Writing $y := \iota(v)(\iota(v)+a)^{-1}$, the above equation says that yw commutes with $p_{n,R} = \begin{pmatrix} 1_{n,R} & 0 \\ 0 & 0_{n,R} \end{pmatrix}$, so it has the form

$$yw = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in \mathrm{GL}_{2n}(R),$$

and of course, $q(yw) = \mathrm{diag}(q(c), q(d))$. But we also have

$$q(yw) = q(\iota(v))q(\iota(v) + a)^{-1}q(w) = q(w) = \mathrm{diag}(u, v).$$

Therefore, $c \in \mathrm{GL}_n(R)$ lifts $u \in \mathrm{GL}_n(R/I)$, thus $q_*[c] = [u]$.

- [$\mathrm{Range}\,\check{\partial} \supseteq \ker(\alpha \circ \tilde{\iota})_*$.] Recall from (12.3) that any $x \in K_0(I)$ may be written as

$$x = [p_n + t] - [p_n], \tag{12.6}$$

where $t \in M_{2n}(I)$. Now suppose $(\alpha \circ \tilde{\iota})_*(x) = 0$ in $K_0(R)$. That is,

$$(\alpha \circ \tilde{\iota})_*(x) = [p_{n,R} + t] - [p_{n,R}] = 0 \in K_0(R).$$

The subsequent argument is adapted from Blackadar's *K-theory for operator algebras.* By increasing n if necessary, we may assume that the conjugacy relation

$$p_{n,R} + t = w p_{n,R} w^{-1}$$

holds, for some $w \in \mathrm{GL}_N(R)$. Writing $W = \mathrm{diag}(w, 1_{N,R}, w^{-1}, 1_{N,R})$, we still have

$$p_{n,R} + t = W p_{n,R} W^{-1}, \qquad W \in \mathrm{E}_{4N}(R),$$

where we suppress explicit mention of extra appended 0s. Modulo I, the above becomes

$$p_{n,R/I} = q(W) p_{n,R/I} q(W)^{-1}.$$

This forces $q(W)$ to have the form $q(W) = \mathrm{diag}(u_1, u_2)$, where $u_1 \in \mathrm{GL}_N(R/I)$ and $u_2 \in \mathrm{GL}_{3N}(R/I)$. As $W \in \mathrm{E}_{4N}(R)$, thus $q(W) \in \mathrm{E}_{4N}(R/I)$, by further increasing N if necessary, we may assume that $\mathrm{diag}(u_1^{-1}, 1_{2N,R/I})u_2^{-1} \in \mathrm{E}_{3N}(R)$ (see the rotation argument in (7.9)). Then there exists $v \in \mathrm{E}_{3N}(R) \subseteq \mathrm{GL}_{3N}(R)$ such that $q(v) = \mathrm{diag}(u_1^{-1}, 1_{2N,R/I})u_2^{-1}$. Set

$$z = \mathrm{diag}(1_{N,R}, v)W \in \mathrm{GL}_{4N}(R),$$

then

$$\begin{aligned} q(z) &= \mathrm{diag}(1_{N,R/I}, u_1^{-1}, 1_{2N,R/I})\mathrm{diag}(1_{N,R/I}, u_2^{-1})\mathrm{diag}(u_1, u_2) \\ &= \mathrm{diag}(u_1, u_1^{-1}, 1_{2N,R}). \end{aligned}$$

Furthermore,

$$zp_nz^{-1} = \mathrm{diag}(1_{N,R}, v)\underbrace{Wp_{n,R}W^{-1}}_{=p_{n,R}+t}\mathrm{diag}(1_{N,R}, v^{-1}) = p_{n,R} + t.$$

(Recall that $p_{n,R} + t$ is actually $(p_{n,R} + t) \oplus 0_{3N,R}$.)

Therefore, by definition,

$$\check{\partial}[u_1] = [zp_nz^{-1}] - [p_n] = [p_n + t] - [p_n] \overset{(12.6)}{=} x.$$

□

If we did not demonstrate the excision property, the it is still possible to directly show the exactness of the upper sequence of Theorem 12.5, based on the definition of $\partial : K_1(R/I) \to K_0(R, I)$ in (12.5). The proof is more or less the same as what we did for the lower sequence with $\check{\partial}$, but we would need to justify starting from the standard form of $x \in K_0(R, I)$ deduced in the excision property proof, Eq. (12.2). So some form of the excision proof would be used anyway.

Since we had shown the excision property for K_0, it is simpler to just use $K_0(I)$ and compute with $\check{\partial}$. This is routinely done, e.g., for *topological K_0-theory*, which coincides with the algebraic K_0 whenever both are defined. However, one also demands that $K_1^{\mathrm{top}}, K_2^{\mathrm{top}}$ etc. satisfy the excision property, and so these higher topological K-functors are defined differently from the algebraic K_1. The algebraic K_1 does not generally satisfy an excision property, so we have to consider $K_1(R, I)$, instead of just, e.g., $K_1(I) = \ker((\pi_{\mathbb{Z}})_* : K_1(\mathbb{Z} \oplus I) \to K_1(\mathbb{Z}))$. Since we were still able to find a natural connecting map $\partial : K_2(R/I) \to K_1(R, I)$, it is instructive to also find $\partial : K_1(R/I) \to K_0(R, I)$, rather than just $\check{\partial} : K_1(R/I) \to K_0(I)$.

12.5 Connecting map as an index map

Let $\mathcal{H}$ be a Hilbert space, and consider $R = \mathcal{L} = \mathcal{L}(\mathcal{H})$ and $I = \mathcal{K} = \mathcal{K}(\mathcal{H})$ the ideal of compact operators. One definition of $\mathcal{K}$ is the operator norm closure of the finite rank operators; some more work is needed to establish the ideal property, and closure under adjoints.

The quotient ring $\mathcal{Q} = \mathcal{L}/\mathcal{K}$ is called the *Calkin algebra*. Because $\mathcal{K}$ is a closed $*$-ideal, $\mathcal{Q}$ is actually a Banach $*$-algebra. (In fact, it is a so-called C^*-algebra.) There is an exact sequence (of non-unital rings),

$$0 \to \mathcal{K} \to \mathcal{L} \xrightarrow{q} \mathcal{Q} \to 0,$$

and we would like to investigate the connecting map

$$\check{\partial} : K_1(\mathcal{Q}) \to K_0(\mathcal{K}) = K_0(\mathcal{K})$$

The formula (12.4) works for any invertible $u \in \mathrm{GL}_n(\mathcal{Q})$ representing a $K_1(\mathcal{Q})$ class.

Now suppose $u \in \mathrm{GL}_n(\mathcal{Q})$ is a *unitary* element, i.e., $u^* = u^{-1}$. Then $\mathrm{diag}(u, u^{-1})$ admits a lift in $\mathrm{GL}_{2n}(\mathcal{L})$, but we can often do better.

Example 12.1. Consider the Hilbert subspace $\iota : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{Z})$. Let S be the left shift operator on $\ell^2(\mathbb{Z})$, and consider the $v = \iota^* \circ S \circ \iota \in \mathcal{L}(\ell^2(\mathbb{N}))$. The operator v is just the unilateral left shift on $\ell^2(\mathbb{N})$. It is no longer a unitary operator, but just a partial isometry, with

$$v^*v = P_{\overline{\mathrm{span}}\{e_n \,:\, n \geq 1\}}, \qquad vv^* = 1_{\ell^2(\mathbb{N})}.$$

Modulo a finite-rank operator, v is a unitary. So under the quotient map $q : \mathcal{L} \to \mathcal{Q}$, we have $q(v)$ being a unitary (which of course lifts to a partial isometry).

Returning to a general unitary $u \in \mathrm{GL}_n(\mathcal{Q})$, suppose u admits a partial isometric lift $v \in M_n(\mathcal{L})$. Then the following is a unitary lift for $\mathrm{diag}(u, u^{-1})$,

$$w = \begin{pmatrix} v & 1 - vv^* \\ 1 - v^*v & v^* \end{pmatrix}.$$

Then, following the formula (12.4), we have

$$\begin{aligned}
\check{\partial}[u] &= [wp_nw^{-1}] - [p_n] \\
&= \left[\begin{pmatrix} vv^* & 0 \\ 0 & 1_n - v^*v \end{pmatrix} \right] - [p_n] \\
&= -[1_n - vv^*] + \underbrace{[1_n - vv^*] + [vv^*]}_{=[1_n] \text{ by Ex. } 12.1} + [1 - v^*v] - [1_n] \\
&= [1 - v^*v] - [1 - vv^*] \\
&\equiv [P_{\ker v}] - [P_{\mathrm{coker}\, v}] \in K_0(\mathcal{K}).
\end{aligned}$$

It may be shown that $K_0(\mathcal{K}) \cong \mathbb{Z}$, with the isomorphism given by the difference of ranks of representative idempotents. (This is actually not an obvious result, requiring some general C^*-algebra theory.) Therefore,

$$\check{\partial}[q(u)] = \check{\partial}[u] = \dim\ker v - \dim\operatorname{coker} v \equiv \operatorname{Index}(v),$$

where the right side is the *Fredholm index*.

In general, an operator $T \in \mathcal{L}$ is *Fredholm* if it is invertible modulo $\mathcal{K}$. The obstruction to genuine invertibility is captured by

$$\dim\ker T, \qquad \dim\operatorname{coker} T,$$

which can be shown to be finite. By definition,

$$\operatorname{Index}(T) = \dim\ker T - \dim\operatorname{coker} T.$$

Writing $T = v|T|$ for the polar decomposition, we can work with $\dim\ker v$ and $\dim\operatorname{coker} v$ instead. So $\check{\partial}(q(v))$ is identified with the Fredholm index of T. This depends only on the K_1 class of $q(v)$.

In analogy to the Fredholm index, the K_1-to-K_0 connecting map is sometimes also called an "algebraic index map".

Remark. Of course, one of the main points of working with the *topological* algebras $\mathcal{K}, \mathcal{L}, \mathcal{Q}$, is to express the *topological* invariance of the Fredholm index. For this, one has to use the topological $K_1^{\text{top}}(-)$, defined in terms of homotopy classes of invertibles (or unitaries). There is a connecting map

$$\partial^{\text{top}} : K_1^{\text{top}}(\mathcal{Q}) \to K_0(\mathcal{K})$$

which takes $[q(v)]_{K_1^{\text{top}}(\mathcal{Q})}$ to the Fredholm index as well.

Generally speaking, K_1^{top} is easier to compute than algebraic K_1, primarily because of the *Bott periodicity* property of topological K-theory, which gives natural equivalences between $K_0(-)$ and $K_{2n}^{\text{top}}(-)$. The latter collapses the long exact sequence in topological K-theory into a cyclic six-term exact sequence, thereby highly constraining the possibilities for the $K_\bullet^{\text{top}}$ groups appearing in the sequence.