6 Infinity categories via simplicial sets

6.1 Homotopy

In **Top**, topological spaces $X, Y \in \mathbf{Top}$ may be related by continuous maps (the (1-)morphisms). At this level, the basic equivalence relation is *homeomorphism* (possessing continuous $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$).

Two morphisms $f_0, f_1 : X \to Y$ are *homotopic* if there exists a morphism

$$f : X \times [0,1] \to Y \tag{6.1}$$

such that $f(-,0) = f_0(-)$ and $f(-,1) = f_1(-)$. It is easy to see that the relation of being homotopic is an equivalence relation.

In particular, if $X = \{\mathrm{pt}\}$ is a singleton, then a morphism $\{\mathrm{pt}\} \to Y$ is specified by a point in Y. So two morphisms $f_0, f_1 : \{\mathrm{pt}\} \to Y$ are homotopic iff there exists a morphism $f : [0,1] \to Y$ with $f(0) = f_0(\mathrm{pt})$ and $f(1) = f_1(\mathrm{pt})$; in other words, if $f_0(\mathrm{pt})$ is path-connected to $f_1(\mathrm{pt})$. Therefore, the set of homotopy classes of maps $\mathrm{pt} \to Y$ coincides with the set of path-components of Y, and is denoted $\pi_0(Y)$.

If X and Y have respective basepoints x, y, then we say that two based maps $f_0, f_1 : (X, x) \to (Y, y)$ are *based homotopic* if there is a homotopy f such that $f(x, t) = y$ for all $t \in [0,1]$ (not just at $t = 0, 1$).

For $X = S^n$ (the n-dimensional sphere with any point as basepoint), the set of based homotopy equivalence classes of maps $(S^n, x) \to (Y, y)$ is denoted $\pi_n(Y, y)$.

- $n = 0$ case. We may write $S^0 = \{\mathrm{pt}, x\}$ with x the basepoint. Then a based map $S^0 \to (Y, y)$ is constrained to take x to y, but it may take pt to any point in Y. So $\pi_0(Y, y)$ does not depend on the basepoint y, and is the same thing as $\pi_0(Y)$.

- $n = 1$ case. Fix a homeomorphism $(S^1, x) \cong [0,1]/_{0\sim 1}$. As discussed in Example 1.1, there is a concatenatation operation on based loops $\ell_1, \ell_2 : (S^1, x) \to (Y, y)$, given by

$$\ell_2 \star \ell_1(t) = \begin{cases} \ell_1(2t), & t \in [0, 1/2], \\ \ell_2(2t-1), & t \in [1/2, 1]. \end{cases}$$

Each based loop $\ell : (S^1, x) \to (Y, y)$ has a reversed version ℓ^{rev}, with

$$\ell^{\text{rev}}(t) = \ell(1-t).$$

The concatenation operation descends to an associative composition rule on the based homotopy classes,

$$[\ell_2] \cdot [\ell_1] = [\ell_2 \star \ell_1] \in \pi_1(Y, y).$$

The constant loop const_y satisfies

$$[\ell] \cdot [\text{const}_y] = [\ell] = [\text{const}_y] \cdot [\ell] \in \pi_1(Y, y),$$

for all based loops ℓ, while $[\ell^{\text{rev}}]$ is the inverse of $[\ell]$. So $\pi_1(Y, y)$ forms a *group*, called the *fundamental group* of (Y, y).

- $n \geq 2$ case. We identify (S^n, x) with $[0,1]^n / \sim$, where $\sim$ identifies all points on the boundary of the cube with a single point. Two based maps $f, g : (S^n, x) \to (Y, y)$ can be "concatenated" by taking

$$(g \star f)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n), & t_1 \in [0, 1/2], \\ g(2t_1 - 1, t_2, \ldots, t_n), & t_1 \in [1/2, 1]. \end{cases}$$

This descends to a composition rule on $\pi_n(Y, y)$, making it an *abelian group*, called the *n-th homotopy group* of (Y, y). (The proof of abelianness involves an argument using the extra coordinate t_2 to "rotate" $f \star g$ to $g \star f$, and is omitted.)

6.1.1 Fundamental groupoid

Observe that if $f, g : [0,1] \to Y$ are paths with $f(1) = g(0)$, then we can still concatenate them into a single path. This suggests all the $\pi_1(Y, y), y \in Y$ should be considered simultaneously.

So, consider the (small) category $\pi_{\leq 1}(Y)$ to have objects being the points of Y, and

$$\text{Hom}_{\pi_{\leq 1}(Y)}(y, y') = \{f : [0,1] \to Y \mid f(0) = y,\ f(1) = y'\}/_{\text{homotopy}}.$$

Here, a homotopy is required to maintain the start point y and the endpoint y'. While path concatenation is not strictly associative, it does descends to an

associative composition rule on homotopy classes of paths,

$$\begin{aligned}\mathrm{Hom}_{\pi_{\leq 1}(Y)}(y_1, y_2) \times \mathrm{Hom}_{\pi_{\leq 1}(Y)}(y_0, y_1) &\to \mathrm{Hom}_{\pi_{\leq 1}(Y)}(y_0, y_2)\\ ([g], [f]) &\mapsto [g \star f],\end{aligned}$$

making $\pi_{\leq 1}(Y)$ a category.

In the category $\pi_{\leq 1}(Y)$, two objects y, y' are isomorphic iff there exists a (homotopy class of) path connecting them. So

$$\pi_0(Y) = \mathrm{Ob}(\pi_{\leq 1}(Y))/_{\text{Isomorphism}}.$$

We also recover the fundamental groups at each base point y as

$$\pi_1(Y, y) = \mathrm{Hom}_{\leq 1}(y, y).$$

Although $\mathrm{Hom}_{\pi_{\leq 1}(Y)}(y, y')$ is not a group when $y \neq y'$, every $[f] \in \mathrm{Hom}_{\pi_{\leq 1}(Y)}(y, y')$ has an inverse $[f^{\mathrm{rev}}] \in \mathrm{Hom}_{\pi_{\leq 1}(Y)}(y', y)$, since

$$\begin{aligned}[f] \circ [f^{\mathrm{rev}}] &= [f \star f^{\mathrm{rev}}] = [\mathrm{const}_{y'}] = \mathrm{id}_{y'}\\ [f^{\mathrm{rev}}] \circ [f] &= [f^{\mathrm{rev}} \star f] = [\mathrm{const}_{y}] = \mathrm{id}_{y}.\end{aligned}$$

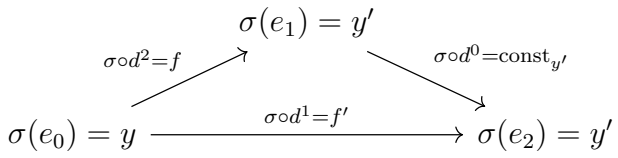
So $\pi_{\leq 1}(Y)$ is a special kind of category, whose morphisms are all invertible, i.e., it is a *groupoid*. We usually call $\pi_{\leq 1}(Y)$ the *fundamental groupoid* of Y.

6.1.2 Simplex viewpoint

It is conceivable that there is a "higher groupoid" version of $\pi_{\leq 1}(Y)$ which can also reproduce all the higher homotopy groups, $\pi_n(Y)$, $n > 1$.

Notice that in $\pi_{\leq 1}(Y)$, we took homotopy classes of paths as the 1-morphisms, but took points (rather than path-components) as objects. Let us restore the actual paths, and reformulate the homotopy relation in terms of singular simplices.

Consider singular 1-simplices f, f' having the same start point y and the same end point y'. A *homotopy* from f to f' is a singular 2-simplex σ whose faces[5] are as depicted below:

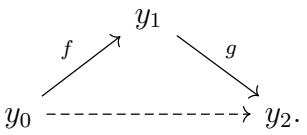


. (6.2)

[5] A singular n-simplex σ has $n+1$ faces $\sigma \circ d^j, 0 \leq j \leq n$, each of which is a singular $(n-1)$-simplex, see Section 3.1.

It is clear that homotopy defines a reflexive and symmetric relation; transitivity is handled later. As an exercise, one can check that f, f' are homotopic in the above sense iff they are homotopic in the original sense (Eq. (6.1), with endpoints fixed).

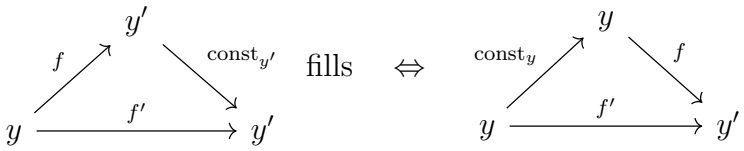
We would like to regard $\mathrm{Sing}_1(Y)$ (the unparametrized paths in Y) as the set of "morphisms" joining various elements of the object set $\mathrm{Sing}_0(Y)$ (the set of points in Y). The quotation marks indicate that "morphism" is not meant in the sense of ordinary categories. As previously mentioned, we have to relax the strict notion of (associative) path-composition. Instead, we allow for different compositions of singular 1-simplices, as witnessed by singular 2-simplices. Specifically, let f be a singular 1-simplex joining y_0 to y_1 and g be a singular 1-simplex joining y_1 to y_2. Let σ be a singular 2-simplex with vertices y_0, y_1, y_2, with its 2-nd face being f and 0-th face being g; pictorially,



(6.3)

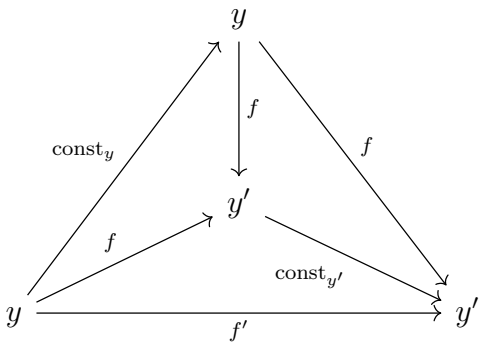
Then the remaining face $y_0 \dashrightarrow y_2$ is regarded as *a* composition gf, witnessed by σ. We do not say "*the* composition", since $y_0 \dashrightarrow y_2$ depends on σ, and different σ can give the same $\dashrightarrow$. It is customary to say that f, g constitute a "horn". Whenever this horn is "filled up" to a 2-simplex, we get a composition of g with f.

Although the definition (6.2) of homotopy seems to prefer y', we actually have the following equivalence:



(6.4)

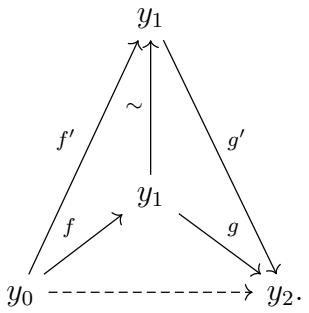
To see why, consider



. (6.5)

The left and right small triangles can be filled by an obvious 2-simplex. If the left side of (6.4) holds, then the bottom small triangle also fills. View the vertices of the diagram as those of a possible 3-simplex. Specifically, the lower y and y' are e_0 and e_3, while the top y is e_1 and the middle y' is e_2. The three small triangles have been filled to form three faces. By the "horn-filling" principle (to be elaborated later) — there exists a singular 3-simplex (tetrahedron) such that three of its faces match those three filled faces. The last face of this 3-simplex fills up the remaining large base triangle. But this last filling is precisely the right side of (6.4). The converse argument is similar.

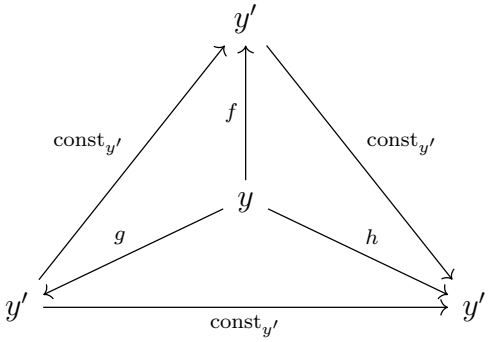
Remark. We can attach the 2-simplices of (6.4) to (6.3),



This suggests that there is a genuine composition rule on homotopy equivalence classes.

A similar horn-filling argument enters when showing that the relation of homotopy is transitive. Suppose f, g, h are singular 1-simplices (with start point y and end point y'), such that f is homotopic to g (so g is homotopic to

f), and g is homotopic to h. Consider the tetrahedral diagram



. (6.6)

The left and bottom small triangles are filled by some 2-simplices by assumption. The large base triangle is filled by the constant 2-simplex with value y'. By the horn-filling argument, there exists a 2-simplex filling the remaining right small triangle. The latter is the statement that f is homotopic to h.

The above story indicates that homotopy theory of Y can be formulated in terms of the simplicial set $\mathrm{Sing}(Y)$. Then, for more general simplicial sets (at least those satisfying appropriate "horn-filling" conditions), there is also an abstract homotopy theory.

6.2 Horns in simplicial sets

If S is a simplicial set, a *simplicial subset* T is a simplicial set such that

$$\begin{aligned} T([n]) &\subseteq S([n]), & n &\geq 0, \\ T(a) &= S(a)|_{T([n])}, & [m] &\xrightarrow{a} [n]. \end{aligned}$$

In this case, we write $T \subseteq S$, and the inclusion $\iota = \{\iota_n : T([n]) \hookrightarrow S([n])\}_{n\geq 0}$ is a natural transformation $\iota : T \Rightarrow S$.

Example 6.1. For $i = 0, \ldots, n$, let $d^i : [n-1] \to [n]$ be the injective (order-preserving) map which misses i. This induces a pushforward natural transformation $(d^i)_* : \Delta^{n-1} \Rightarrow \Delta^n$. The image of $(d^i)_*$ is a simplicial subset of Δ^n, denoted $\Delta^{[n]\setminus\{i\}}$. Explicitly,

$$\begin{aligned} \Delta^{[n]\setminus\{i\}}([m]) &= \big\{\text{order-preserving maps } [m] \to [n] \setminus \{i\}\big\} \subseteq \Delta^n([m]), \\ \Delta^{[n]\setminus\{i\}}\big([m] \xrightarrow{a} [m']\big) &\mapsto a^*|_{\Delta^{[n]\setminus\{i\}}([m'])}. \end{aligned}$$

It is customary to call $\Delta^{[n]\setminus\{i\}}$ the i-th *face* of Δ^n, and $(d^i)_* : \Delta^{n-1} \Rightarrow \Delta^n$ is the "inclusion map" for the i-th face. This terminology is justified by the fact that under $\mathrm{GR} : \Delta \to \mathbf{Top}$ (geometric realization for representables),

$$\begin{aligned} \big(d^i : [n-1] \to [n]\big) &\in \mathrm{Hom}_\Delta([n-1],[n]) \\ &\overset{\mathrm{GR}}{\mapsto} \quad \big(\widetilde{d^i} : |\Delta^{n-1}| \to |\Delta^n|\big) \in \mathrm{Hom}_{\mathbf{Top}}(|\Delta^{n-1}|, |\Delta^n|), \end{aligned}$$

where $\widetilde{d^i}$ are the topological face-inclusion maps of (3.1). As explained in Section 5.3.1, this GR is understood as a special case of $|\cdot| : \mathbf{Set}_\Delta \to \mathbf{Top}$, so

$$\begin{aligned} \big((d^i)_* : \Delta^{n-1} \Rightarrow \Delta^n\big) &\in \mathrm{Hom}_{\mathbf{Set}_\Delta}(\Delta^{n-1}, \Delta^n) \\ &\overset{|\cdot|}{\mapsto} \quad \big(\widetilde{d^i} : |\Delta^{n-1}| \to |\Delta^n|\big) \in \mathrm{Hom}_{\mathbf{Top}}(|\Delta^{n-1}|, |\Delta^n|). \end{aligned}$$

Example 6.2. More generally, each subset $K \subseteq [n]$ corresponds to a simplicial subset $\Delta^K \subseteq \Delta^n$, whose m-simplices are those order-preserving maps $[m] \to [n]$ which land in K. Writing $|K| = k+1$, there is a unique injective order-preserving map $d^K : [k] \to [n]$ which lands on K, and it induces the natural transformation $(d^K)_* : \Delta^k \Rightarrow \Delta^n$ whose image is Δ^K. Under GR, d^K becomes the injective placement map $\widetilde{d^K} : |\Delta^k| \to |\Delta^n|$ (see Eq. (3.3)), whose image is the subsimplex Δ^K_{top} with vertex set $\{e_i\}_{i\in K}$.

We can factorize $(d^K)_*$ as $\Delta^k \overset{\mathrm{iso.}}{\Rightarrow} \Delta^K \overset{\mathrm{incl.}}{\Rightarrow} \Delta^n$, and correspondingly, $\widetilde{d^K}$ factorizes as $|\Delta^k| \overset{\cong}{\to} \Delta^K_{\mathrm{top}} \hookrightarrow |\Delta^n|$. As in Section 5.3.1, we identify Δ^K_{top} as the geometric realization $|\Delta^K|$, then $|\Delta^K| \hookrightarrow |\Delta^n|$ is identified with the geometric realization of the inclusion $\Delta^K \Rightarrow \Delta^n$ of simplicial sets. Furthermore, if $K \subseteq K' \subseteq [n]$, then the inclusion $\Delta^K \Rightarrow \Delta^{K'}$ is geometrically realized as the inclusion $|\Delta^K| \hookrightarrow |\Delta^{K'}|$.

Example 6.3. There are other important simplicial subsets of Δ^n which are not of the form Δ^K. For $0 \le j \le n$, the *j-th horn* of Δ^n is the simplicial subset

$$\Lambda^n_j = \bigcup_{0\le i\le n,\, i\ne j} \Delta^{[n]\setminus\{i\}}.$$

Explicitly, the m-simplices in Λ^n_j are

$$\Lambda^n_j([m]) = \big\{[m] \overset{a}{\to} [n] \,:\, \mathrm{Image}(a) \cup \{j\} \ne [n]\big\}. \tag{6.7}$$

Geometric realization of $\Lambda^n_j \Rightarrow \Delta^n$ is the inclusion $|\Lambda^n_j| \to |\Delta^n|$, where $|\Lambda^n_j|$ is the union of all the faces of $|\Delta^n|$ except for the j-th one. So $|\Lambda^n_j|$ is shaped like a "horn" with the vertex e_j being the "mouthpiece" of the horn.

Remark. It takes a bit of work to fully justify the above description of $|\Lambda^n_j|$.

- By definition, $|\Lambda^n_j|$ is a colimit object, associated to how Λ^n_j is a colimit of representables. We had used the "tautological" presentation of the colimit, but there are other ways to exhibit Λ^n_j as a colimit of representables.

- Let $\mathcal{I}$ be the indexing category whose objects are subsets $K \subseteq [n]$ such that $\Delta^K \subseteq \Lambda^n_j$. Inspecting (6.7), only the subsets $[n] \setminus \{j\}$ and $[n]$ are excluded from $\mathcal{I}$. Inclusions $K \subseteq K'$ correspond to arrows in $\mathcal{I}$.

- Consider the diagram $D : \mathcal{I} \to \mathbf{Set}_\Delta$, taking K to Δ^K and $K \subseteq K'$ to $\Delta^K \overset{\iota}{\Rightarrow} \Delta^{K'}$. According to Theorem 4.4, $\varinjlim D$ can be computed objectwise, so we investigate

$$\varinjlim \mathrm{ev}_{[m]} \circ D = \varinjlim_{K \in \mathrm{Ob}(\mathcal{I})} \Delta^K([m]) = \varinjlim_{K \in \mathrm{Ob}(\mathcal{I})} \Delta^K([m]), \qquad m \geq 0.$$

  The set $\bigcup_{K \in \mathcal{I}} \Delta^K([m])$ is a cone for the diagram $\mathrm{ev}_{[m]} \circ D$, and we check that it is a colimit (Exercise). In other words,

$$\Lambda^n_j([m]) = \bigcup_{K \in \mathrm{Ob}(\mathcal{I})} \Delta^K([m]) = \varinjlim \mathrm{ev}_{[m]} \circ D.$$

  Then

$$\Lambda^n_j = \bigcup_{K \in \mathrm{Ob}(\mathcal{I})} \Delta^K = \varinjlim D.$$

  Note that Δ^K are representable (naturally isomorphic to Δ^k), so we have exhibited Λ^n_j as a colimit of representables in a different way.

- In assignments, we saw that right-adjoint functors preserve limits[6]; Dually, left-adjoint functors preserve colimits. As geometric realization is a left-adjoint functor,

$$|\Lambda^n_j| = |\varinjlim D| = \varinjlim |D| = \varinjlim_{K \in \mathrm{Ob}(\mathcal{I})} |\Delta^K|.$$

  One checks that the last colimit in **Top** can be taken to be the union of all the $|\Delta^K|$. Recalling that K runs over all subsets of $[n]$ except $[n] \setminus \{j\}$ and $[n]$, we see that $|\Lambda^n_j|$ is the union of all the faces of $|\Delta^n|$ except the j-th face.

[6] Actually the limit cone structure is preserved, not just the limiting object.

6.2.1 Horn-filling for $\mathrm{Sing}(X)$

Proposition 6.1. *Let X be a topological space. For every $n \geq 1$, $0 \leq j \leq n$, any $\Lambda^n_j \Rightarrow \mathrm{Sing}(X)$ admits an extension to $\Delta^n \Rightarrow \mathrm{Sing}(X)$.*

Proof. Because of the adjunction $|\cdot| \dashv \mathrm{Sing}$, we have bijections

$$\overline{(\cdot)} : \mathrm{Hom}_{\mathbf{Set}_\Delta}(S, \mathrm{Sing}(X)) \cong \mathrm{Hom}_{\mathbf{Top}}(|S|, X),$$

depending naturally on S. Given $f : \Lambda^n_j \Rightarrow \mathrm{Sing}(X)$, we have $\overline{f} : |\Lambda^n_j| \to X$. Note that $|\Lambda^n_j|$ is a *retract* of $|\Delta^n|$, i.e., there is a continuous map $r : |\Delta^n| \to |\Lambda^n_j|$ which restricts to the identity on $|\Lambda^n_j|$. So we can form a commutative diagram

$$\begin{array}{ccc} |\Lambda^n_j| & \xrightarrow{\overline{f}} & X \\ {\scriptstyle |\iota|}\downarrow & \nearrow_{\overline{f}\circ r} & \\ |\Delta^n| & & \end{array} .$$

(This explains the existence of the singular 3-simplices used in Section 6.1.2.) Using the natural adjunction bijections, we get a corresponding commutative diagram

$$\begin{array}{ccc} \Lambda^n_j & \xRightarrow{f} & \mathrm{Sing}(X) \\ {\scriptstyle \iota}\Downarrow & \nearrow_{\tilde{f}} & \\ \Delta^n & & \end{array} ,$$

where $\tilde{f}$ is the unique element of $\mathrm{Hom}_{\mathbf{Set}_\Delta}(\Delta^n, \mathrm{Sing}(X))$ such that $\overline{\tilde{f}} = \overline{f} \circ r$. This $\tilde{f}$ is an extension of f. □

6.2.2 General horn-filling problem

Let S be a simplicial set. For each order-preserving map $[m] \xrightarrow{a} [n]$, we have an induced function $S(a) : S([n]) \to S([m])$. In particular, for the face maps $d^i : [n-1] \to [n]$ which skips i (where $0 \leq i \leq n$), the induced functions are denoted

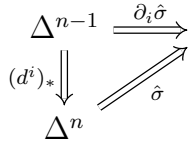
$$\partial_i \equiv S(d^i) : S([n]) \to S([n-1]).$$

For an n-simplex $\sigma \in S([n])$, we call $\partial_i \sigma \in S([n-1])$ its i-th face. This terminology is justified by Remark 3 below.

Remark 3. By the Yoneda Lemma, $\sigma \in S([n])$ corresponds to a unique natural transformation $\hat{\sigma} : \Delta^n \Rightarrow S$, characterized by

$$\hat{\sigma}_{[n]}(\mathrm{id}_{[n]}) = \sigma.$$

Let us restrict $\hat{\sigma}$ to the i-th face of Δ^n. More precisely, this is the natural transformation $\partial_i\hat{\sigma} : \Delta^{n-1} \Rightarrow S$ which makes the diagram below commute,



. (6.8)

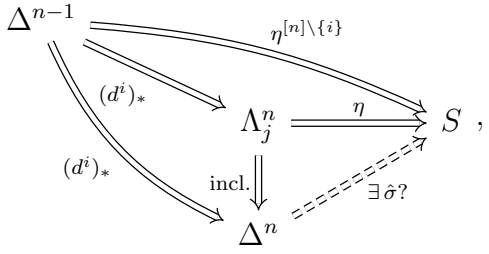
Here, $(d^i)_*$ is the inclusion map for the i-th face (see Example 6.1). By the Yoneda Lemma, $\partial_i\hat{\sigma}$ is characterized by what it does to $\mathrm{id}_{[n-1]}$,

$$\begin{aligned}
(\partial_i\hat{\sigma})_{[n-1]}(\mathrm{id}_{[n-1]}) &= \hat{\sigma}_{[n-1]}((d^i)_*(\mathrm{id}_{[n-1]})) && \text{(Eq. (6.8))} \\
&= \hat{\sigma}_{[n-1]}(d^i) \\
&= S(d^i)(\sigma) && \text{(Yoneda formula (2.5))} \\
&= \partial_i\sigma.
\end{aligned} \tag{6.9}$$

Thus, under the Yoneda bijection, the restriction of $\hat{\sigma}$ to the i-th face $\partial_i\hat{\sigma}$ of Δ^n corresponds to what we called the "i-th face" $\partial_i\sigma$ of σ.

Thus, we will generally draw an n-simplex $\sigma : \Delta^n \Rightarrow S$ just as a simplex Δ^n (or rather, $|\Delta^n|$), implicitly understood to be "placed in S". The faces of this drawing are then the faces $\partial_j\sigma$, again, implicitly understood to be placed in S. Similarly, horns in S are drawn just as a (geometrically realized) horn.

Suppose we have a horn $\eta : \Lambda^n_j \Rightarrow S$. The *horn-filling problem* asks for the existence of an n-simplex $\hat{\sigma} : \Delta^n \Rightarrow S$ extending η. In particular, the following diagram should commute,



, $0 \le i \le n,\ i \neq j.$ (6.10)

In view of (6.8), $\hat{\sigma}$ solves the horn-filling problem if

$$\partial_i \hat{\sigma} = \eta^{[n]\setminus\{i\}}, \qquad \forall\, 0 \le i \le n,\, i \ne j, \tag{6.11}$$

holds.

A simplicial set S is said to have the Λ^n_j-*horn-filling property*, if all horns $\Lambda^n_j \Rightarrow S$ can be filled to become simplices $\Delta^n \Rightarrow S$. If S has the horn-filling property for all horns $\Lambda^n_j \Rightarrow S$ $(0 \le j \le n,\, n \ge 1)$, it is called a *Kan complex*. Proposition 6.1 says that $\mathrm{Sing}(X)$ are Kan complexes.

6.2.3 Nerves have inner horn-filling

For $n \ge 2$, horns Λ^n_j with $0 < j < n$ are called *inner horns*. Let us study another important class of simplicial sets which only have the horn-filling property for inner horns.

Recall the nerve $N(\mathcal{C})$ of a small category $\mathcal{C}$. This is a simplicial set whose set of abstract n-simplices is (Yoneda identified) with the set of n-arrow-compositions in $\mathcal{C}$,

$$c_0 \overset{f_1}{\to} c_1 \overset{f_2}{\to} \dots c_{n-1} \overset{f_n}{\to} c_n.$$

By convention, $N(\mathcal{C})([0]) = \mathrm{Ob}(\mathcal{C})$. Given an order-preserving map $a : [m] \to [n]$, the function $N(\mathcal{C})(a) : N(\mathcal{C})([n]) \to N(\mathcal{C})([m])$ is obtained by suitably composing some of the f_i, or inserting extra identity morphisms. For example, the face maps $d^j : [n-1] \to [n], 0 \le j \le n$, which skip j, induce the face functions

$$\partial_j \equiv N(\mathcal{C})(d^j) : N(\mathcal{C})([n]) \to N(\mathcal{C})([n-1])$$

$$\big(c_0 \overset{f_1}{\to} \dots \overset{f_n}{\to} c_n\big) \mapsto \begin{cases} \big(c_0 \overset{f_1}{\to} \dots \overset{f_{j-1}}{\to} c_{j-1} \overset{f_{j+1}\circ f_j}{\to} c_{j+1} \overset{f_{j+2}}{\to} \dots \overset{f_n}{\to} c_n\big), & 0 < j < n, \\ \big(c_1 \overset{f_2}{\to} \dots \overset{f_n}{\to} c_n\big), & j = 0, \\ \big(c_0 \overset{f_1}{\to} \dots \overset{f_{n-1}}{\to} c_{n-1}\big), & j = n. \end{cases} \tag{6.12}$$

In particular, a 1-simplex $\big(c_0 \overset{f}{\to} c_1\big) \in N(\mathcal{C})([1])$ has faces

$$\partial_0\big(c_0 \overset{f}{\to} c_1\big) = c_1, \qquad \partial_1\big(c_0 \overset{f}{\to} c_1\big) = c_0,$$

while a 2-simplex $\sigma = \big(c_0 \overset{f_1}{\to} c_1 \overset{f_2}{\to} c_2\big) \in N(\mathcal{C})([2])$ has faces

$$\partial_0\sigma = \big(c_1 \overset{f_2}{\to} c_2\big), \qquad \partial_1\sigma = \big(c_0 \overset{f_2\circ f_1}{\to} c_2\big), \qquad \partial_2\sigma = \big(c_0 \overset{f_1}{\to} c_1\big).$$

Suppose we are given a horn in $N(\mathcal{C})$,

$$\eta : \Lambda^2_1 = \big(\Delta^{\{1,2\}} \cup \Delta^{\{0,1\}}\big) \Rightarrow N(\mathcal{C}).$$

- The horn η restricts to natural transformations

  $$\Delta^{\{1,2\}} \Rightarrow N(\mathcal{C}), \qquad \Delta^{\{0,1\}} \Rightarrow N(\mathcal{C}).$$

  Precomposing with $d^0_* : \Delta^1 \Rightarrow \Delta^{\{1,2\}}$ or $d^2_* : \Delta^1 \Rightarrow \Delta^{\{0,1\}}$, respectively, gives the 1-simplices

  $$\eta^{\{0,1\}} : \Delta^1 \Rightarrow N(\mathcal{C}), \qquad \eta^{\{1,2\}} : \Delta^1 \Rightarrow N(\mathcal{C}),$$

  whose respective Yoneda images in $N(\mathcal{C})([1])$ are denoted

  $$\big(c_0 \xrightarrow{f} c_1\big), \qquad \big(c'_1 \xrightarrow{g} c_2\big).$$

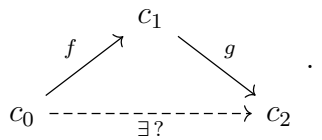
- We may ask whether there exists a 2-simplex $\hat{\sigma} : \Delta^2 \Rightarrow N(\mathcal{C})$ extending $\eta : \Lambda^2_1 \Rightarrow N(\mathcal{C})$. This is an *inner* horn extension problem. According to (6.11) of Section 6.2.2, we require

  $$\partial_0\hat{\sigma} \equiv \hat{\sigma} \circ d^0_* = \eta^{\{1,2\}} \equiv \widehat{g},$$
  $$\partial_2\hat{\sigma} \equiv \hat{\sigma} \circ d^2_* = \eta^{\{0,1\}} \equiv \widehat{f}.$$

  As explained around (6.9), this is equivalent to asking that the Yoneda images $\sigma \in N(\mathcal{C})([2])$ and $f, g \in N(\mathcal{C})([1])$ are related by

  $$\partial_0\sigma = \big(c_1 \xrightarrow{g} c_2\big), \qquad \partial_2\sigma = \big(c_0 \xrightarrow{f} c_1\big). \tag{6.13}$$

  Pictorially, we draw the desired σ as a topological 2-simplex with f on the 2-nd face and g on the 0-th face,

  The 1-st dotted face $c_0 \xrightarrow{\exists ?} c_2$ means that it becomes available once the a σ satisfying (6.13) is found.

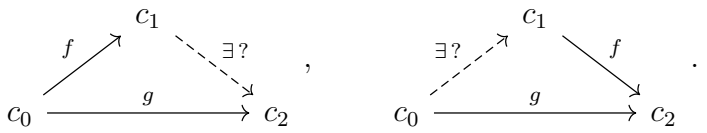
By definition of nerve, $N(\mathcal{C})([2])$ contains the element $\sigma = \big(c_0 \xrightarrow{f} c_1 \xrightarrow{g} c_2\big)$, and this uniquely solves (6.13) holds. Then $\hat{\sigma} : \Delta^2 \Rightarrow N(\mathcal{C})$ is the corresponding *unique* 2-simplex which extends $\eta|_{\Delta^{\{1,2\}}}$ and $\eta|_{\Delta^{\{0,1\}}}$.

Now, $\eta : \Lambda^2_1 \Rightarrow N(\mathcal{C})$ is the unique[7] horn extending $\eta|_{\Delta^{\{1,2\}}}$ and $\eta|_{\Delta^{\{0,1\}}}$. Therefore, $\hat{\sigma}|_{\Lambda^2_1}$ is precisely the horn η. Thus $\hat{\sigma}$ is the unique 2-simplex extending the horn η.

We mention that the dashed arrow $\big(c_0 \xrightarrow{\partial_1\sigma} c_2\big)$ furnished by σ is, by definition (6.12), the composition $g \circ f$ in the category $\mathcal{C}$.

- For higher-dimensional horns ($n \geq 2$), one similarly shows that $N(\mathcal{C})$ satisfies the *inner* horn filling property, and *uniquely* so.

- Similarly, one can ask whether the *outer* horn-filling problems can be solved,



  In general, the answer is no, already for the $n = 2$ horns: Consider $g = \mathrm{id}_{c_0}$, the horn-filling property amounts to the existence of f^{-1} for every morphism f in $\mathcal{C}$. Unless $\mathcal{C}$ is a groupoid, we cannot guarantee this.

6.3 ∞-categories and homotopy category

The notions of horns and horn-filling make sense for any simplicial set S. We have seen that $\mathrm{Sing}(X)$ has (non-unique) horn-filling, while $N(\mathcal{C})$ has (unique) *inner* horn filling. The following class of simplicial sets encompasses both of these cases.

Definition 1. An ∞-category is a simplicial set which has (not necessarily unique) inner horn filling.

[7] Because each $\eta_{[m]} : \Lambda^2_1([m]) \to N(\mathcal{C})([m])$ has domain $\Lambda^2_1([m]) = \Delta^{\{1,2\}}([m]) \cup \Delta^{\{0,1\}}([m])$.

There are apparently many definitions of ∞-categories, and when given as in Definition 1, ∞-categories are often called *quasicategories*. The main contributors to quasicategory theory are Boardman–Vogt, Joyal, Lurie, and subsequently, many others.

6.3.1 Objects and morphisms in ∞-categories

Since ordinary (small) categories embed into ∞-categories via the nerve construction, we shall also use the symbol $\mathcal{C}$ to denote an ∞-category. Generalizing the case of the nerve of an ordinary category, we shall call the elements of $\mathcal{C}([0])$ "objects", and elements of $\mathcal{C}([1])$ "morphisms" in $\mathcal{C}$. However, for a general ∞-category, the "morphisms" do not have a strict composition law. Nevertheless, we will see that the inner horn-filling property allows for composition-up-to-homotopy.

Recall the face maps $d^j : [n-1] \to [n]$, and the induced face functions

$$\partial_j := \mathcal{C}(d^j) : \mathcal{C}([n]) \to \mathcal{C}([n-1]), \qquad 0 \leq j \leq n.$$

In particular, a morphism $f \in \mathcal{C}([1])$ has $\partial_1(f), \partial_0(f) \in \mathcal{C}([0])$, respectively called the *source* and *target* objects of f. Pictorially, we write f as

$$c \xrightarrow{f} c', \qquad c = \partial_1(f),\ c' = \partial_0(f).$$

Recall the degeneracy map $s^0 : [1] \to [0]$ taking $0, 1 \mapsto 0$. Then

$$\mathcal{C}(s^0) : \mathcal{C}([0]) \to \mathcal{C}([1]).$$

For example, if $\mathcal{C} = \mathrm{Sing}(X)$, then $\mathcal{C}(s^0)(c)$ is the constant path const_c (work this out). Analogously, for any object $c \in \mathcal{C}([0])$, we write

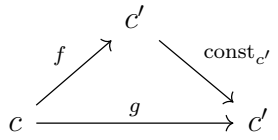
$$\mathrm{const}_c := \mathcal{C}(s^0)(c) \ \in \mathcal{C}([1]),$$

as a distinguished morphism whose source and target are both c,

$$\partial_j(\mathrm{const}_c) = \mathcal{C}(d^j)\mathcal{C}(s^0)(c) = \mathcal{C}(s^0 \circ d^j)(c) = \mathcal{C}(\mathrm{id}_{[0]})(c) = c, \qquad j = 0, 1.$$

As the notation suggests, const_c is meant to be the identity morphism for c, *up to homotopy*.

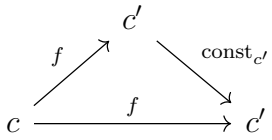
The notion of homotopy generalizes that for $\mathrm{Sing}(X)$ given in (6.2). Namely, $f, g : c \to c'$ are homotopic iff there exists a 2-simplex $\sigma \in \mathcal{C}([2])$ filling[8]



. (6.14)

Unlike the case of $\mathcal{C} = \mathrm{Sing}(X)$, it is not completely obvious that homotopy is reflexive and symmetric. The arguments showing that homotopy is an equivalence relation are:

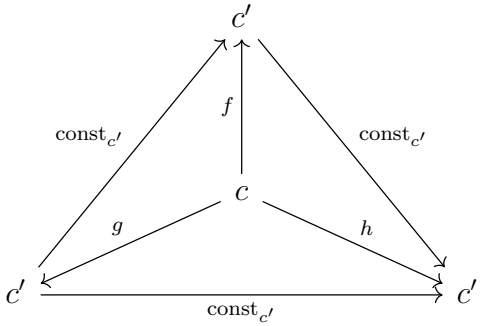
- For each morphism $f \in \mathcal{C}([1])$, there is a 2-simplex $\mathcal{C}(s^1)(f)$, where $s^1 : [2] \to [1]$ is the degeneracy map taking $0 \mapsto 0$ and $1, 2 \mapsto 1$. We check that
$$\partial_j(\mathcal{C}(s^1)(f)) = \begin{cases} \mathrm{const}_{c'}, & j = 0, \\ f, & j = 1, 2, \end{cases}.$$
This means that



fills, (6.15)

thus f is homotopic to f.

- Consider the analogous tetrahedral diagram to (6.6),



. (6.16)

[8]Another interpretation of a 2-simplex σ filling (6.14) is that it is a "2-morphism" joining c to c'.

We use the notation $\sim$ for "homotopic to". We claim that

$$g \sim f \text{ and } g \sim h \;\Rightarrow\; f \sim h.$$

By assumption, the left and bottom triangles are fillable by 2-simplices. The large base triangle is fillable by the "constant 2-simplex at c'". So we have an inner horn Λ^3_1. By the inner horn-filling property, we get a 3-simplex whose first face fills the right small triangle of (6.17). This latter filling is the statement that $f \sim h$.

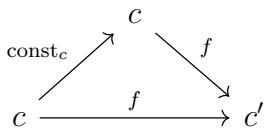
Now specialize to $h = g$, and use reflexivity $g \sim g$ to deduce that[9]

$$g \sim f \;\Rightarrow\; f \sim g. \qquad \text{(symmetry)}.$$

Finally, use symmetry to rewrite the claim as

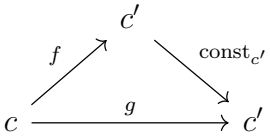
$$f \sim g \text{ and } g \sim h \;\Rightarrow\; f \sim h. \qquad \text{(transitive)}$$

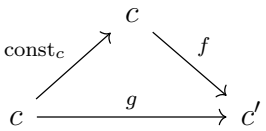
Exercise 6.1. The 2-simplex $\mathcal{C}(s^0)(f)$ shows that



fills. (6.17)

Use the analogous tetrahedral diagram to (6.5), symmetry, and inner-horn filling to show that



fills $\Leftrightarrow$


fills.

Definition 2. Let $\mathcal{C}$ be an ∞-category. For $c, c' \in$, we denote the set of homotopy equivalence classes of morphisms $\left(c \xrightarrow{f} c'\right)$ by $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(c, c')$.

[9]In the case $h = g$, another interpretation of a 3-simplex σ filling (6.17) is a "3-morphism" joining c to (the constant 2-simplex at) c'. It witnesses a "composition" of the left 2-morphism $g \sim f$ and the right 2-morphism $f \sim g$, yielding the bottom 2-morphism $g \sim g$. (Recall Footnote 8, interpreting homotopies as 2-morphisms.) The 2-morphism $g \sim g$ may be regarded as the higher-homotopy class of the "constant" 2-morphism $\mathcal{C}(s^2)(g)$, generalizing (6.19). Thus, in an ∞-category, every 2-morphism has an inverse 2-morphism, up to a 3-morphism.

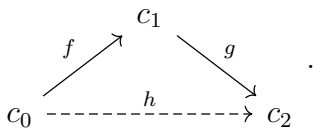
By the following Proposition 6.2, there is a well-defined composition law

$$\begin{aligned} \circ : \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(c_1, c_2) \times \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(c_0, c_1) &\to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(c_0, c_2) \\ ([g], [f]) &\mapsto [h], \end{aligned} \tag{6.18}$$

where h is any possible composition of g and f.

Proposition 6.2. *Let $\mathcal{C}$ be an ∞-category, and $c_0 \xrightarrow{f} c_1$, $c_1 \xrightarrow{g} c_2$ be morphisms in $\mathcal{C}$. Then*

1. *There exists a 2-simplex $\sigma \in \mathcal{C}([2])$ whose 0-th face is g and 2-nd face is f,*



   We say that the 1-st face, $c_0 \xrightarrow{h} c_2$, is a possible composition of g and f, witnessed by σ.

2. *Let $c_0 \xrightarrow{h} c_2$ be a possible composition of g and f as above. Then $c_0 \xrightarrow{h'} c_2$ is another possible composition of g with f iff h and h' are homotopic.*

3. *Let $c_0 \xrightarrow{f'} c_1$, $c_1 \xrightarrow{g'} c_2$, with f homotopic to f' and g homotopic to g'. If h is a possible composition of g and f, and h' is a possible composition of g' and f', then h and h' are homotopic.*

Proof. Exercise. Part of this Proposition was considered in Remark 6.1.2. □

For the composition law (6.18), the (homotopy class of the) constant morphism const_c satisfies

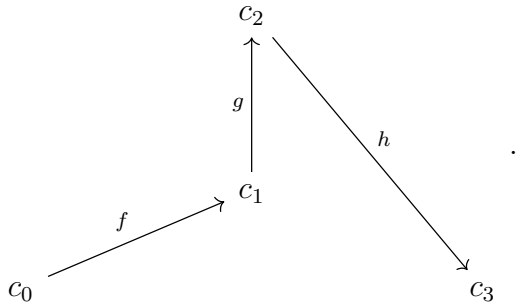
$$[f] \circ [\mathrm{const}_c] = [f] = [f] \circ [\mathrm{const}_{c'}], \qquad \forall\, c \xrightarrow{f} c', \tag{6.19}$$

thus it represents the "identity morphism" for c. This is because the 2-simplices exhibiting $f \sim f$ (Eq. (6.15)-(6.16)) also show that f is a possible composition of const_c and f, and a possible composition of f and $\mathrm{const}_{c'}$.

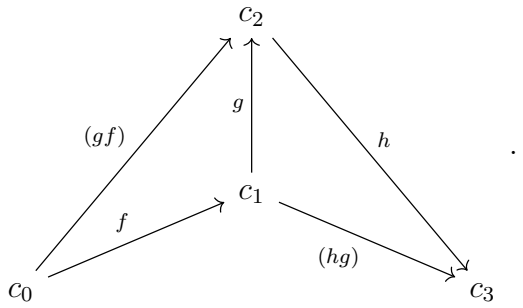
Proposition 6.3. *The composition law* (6.18) *is associative.*

Consider the following three morphisms,

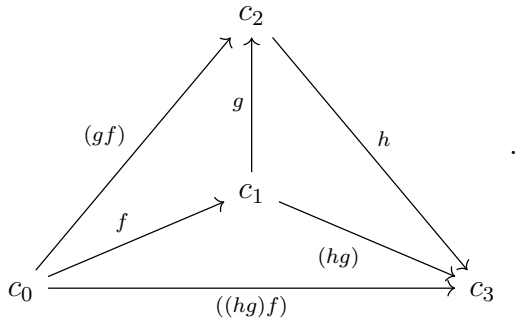
Proof.



As we are working in an ∞-category $\mathcal{C}$, there exists a possible composition of g and f, denoted (gf), as well as a possible composition of h and g, denoted (hg). Diagramatically, we have



Next, there also exists a possible composition of (hg) and f, denoted $((hg)f)$, so we get



At the level of homotopy classes, this shows that

$$\big([h] \circ [g]\big) \circ [f] = [(hg)] \circ [f] = [((hg)f)].$$

Now, the last diagram is an inner horn Λ^3_1, which can be filled to a 3-simplex because $\mathcal{C}$ is an ∞-category. This 3-simplex has a base 2-simplex, witnessing

that $((hg)f)$ is simultaneously a possible composition of h with (gf). Therefore,

$$[((hg)f)] = [h] \circ [(gf)] = [h] \circ \big([g] \circ [f]\big)$$

holds as well. □

6.3.2 Homotopy category

By Proposition 6.3, we can make the following definition.

Definition 3. Let $\mathcal{C}$ be an ∞-category. Its *homotopy category*, denoted $\mathrm{h}\mathcal{C}$, is the (ordinary) category with

- $\mathrm{Ob}(\mathrm{h}\mathcal{C}) = \mathcal{C}([0])$;
- Hom-sets for $c, c' \in \mathrm{Ob}(\mathrm{h}\mathcal{C})$ being $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(c, c')$, with composition law (6.18) and identity morphisms $[\mathrm{const}_c] \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(c, c)$.

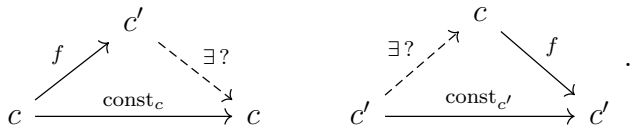
Example 6.4. For a topological space X, the simplicial set $\mathrm{Sing}(X)$ is a Kan complex, thus an ∞-category. Its homotopy category $\mathrm{hSing}(X)$ can be identified with the fundamental groupoid $\pi_{\leq 1}(X)$.

Homotopy inverse. Let $f : c \to c'$ be a morphism in $\mathcal{C}$. A morphism $g : c' \to c$ is a *homotopy inverse* of f if const_c is a possible composition of g and f, and $\mathrm{const}_{c'}$ is a possible composition of f and g. Equivalently,

$$[f] \circ [g] = [\mathrm{const}_{c'}], \qquad [g] \circ [f] = [\mathrm{const}_c] \qquad \text{in } \mathrm{h}\mathcal{C}.$$

If they exist, all homotopy inverses of f are homotopic to each other.

Unlike the case of 2-morphisms (or higher), as sketched in Footnote 9, a (1-)morphism $f : c \to c'$ may not possess any homotopy inverse, because its existence boils down to the *outer* horn-filling problem,



Suppose these fillings exist, so we have morphisms $g, \hat{g} : c' \to c$ such that

$$[f] \circ [\hat{g}] = [\mathrm{const}_{c'}], \qquad [g] \circ [f] = [\mathrm{const}_c].$$

Using associativity, we have

$$[g] = [g] \circ [\mathrm{const}_{c'}] = [g] \circ \big([f] \circ [\hat{g}]\big) = \big([g] \circ [f]\big) \circ [\hat{g}] = [\mathrm{const}_c] \circ [\hat{g}] = [\hat{g}].$$

Then $[f] \circ [g] = [f] \circ [\hat{g}] = [\mathrm{const}_{c'}]$ as well, so g (or $\hat{g}$) is a homotopy inverse of f.

By the above discussion, we have a simple observation:

Proposition 6.4. *Let $\mathcal{C}$ be a Kan complex. Then every morphism in $\mathrm{h}\mathcal{C}$ is invertible (so it is a groupoid).*

Kan complexes are sometimes called $(\infty, 0)$-categories, indicating that all n-morphisms, $n > 0$ are invertible up to $(n+1)$-morphisms. More generally, an ∞-category is also called an $(\infty, 1)$-category, indicating that invertibility only holds for n-morphisms with $n > 1$.

Prop. 6.4 says that the $\mathrm{h}\mathcal{C}$ of a Kan complex $\mathcal{C}$ generalizes the *fundamental groupoid* (special case $\mathcal{C} = \mathrm{Sing}(X)$). Of course, the Kan complex $\mathrm{Sing}(X)$ itself carries more information than $\mathrm{hSing}(X)$ — the latter does not capture $\pi_n(X)$ for $n \geq 2$. In fact, we can recover all the homotopy groups $\pi_n, n \geq 0$ from $\mathrm{Sing}(X)$ (much more theory is needed to formulate this statement). So sometimes, one sees the statement that $\mathrm{Sing}(X)$ is the "fundamental ∞-groupoid" of X, carrying the information of homotopies between paths, homotopies between homotopies between paths, etc.

A modern viewpoint is that Kan complexes are[10] "∞-groupoids" generalizing the fundamental ∞-groupoids $\mathrm{Sing}(X)$, on which a good homotopy theory can be developed. An example of a Kan complex which is not of the form $\mathrm{Sing}(X)$, is the nerve of a groupoid.

[10] This is a definition or a hard theorem depending on what one means by "∞-groupoid".