5 Geometric realization and Singular functor

Recall from (3.4) that we have the functor

$$\mathrm{GR} : \Delta \to \mathbf{Top} \qquad [n] \mapsto |\Delta^n|, \qquad \big([n] \xrightarrow{a} [n']\big) \mapsto |\Delta^n| \xrightarrow{\tilde{a}} |\Delta^{n'}|,$$

which "geometrically realizes" objects of Δ as topological simplices. In view of the Yoneda embedding $h_{(-)} : \Delta \to \mathbf{Set}_\Delta$, we would like to extend GR in the sense that

$$\begin{array}{ccc} \mathbf{Set}_\Delta & & \\ {\scriptstyle h_{(-)}}\uparrow & \overset{|\cdot|?}{\searrow} & \\ \Delta & \xrightarrow{\mathrm{GR}} & \mathbf{Top}. \end{array} \tag{5.1}$$

commutes (up to natural isomorphism). The general procedure of "Yoneda extension" achieves this: take $|\cdot|$ to be the left adjoint functor to

$$\begin{aligned} \mathrm{Sing} : \mathbf{Top} &\to \mathbf{Set}_\Delta \\ X &\mapsto \mathrm{Hom}_{\mathbf{Top}}(\mathrm{GR}(-), X) \\ [n] &\mapsto \mathrm{Sing}_n(X). \end{aligned}$$

While this is very slick (if you are prepared to accept some results in category theory as a black box), it is instructive to be more explicit about the construction of $|\cdot|$.

Here, we mention that we have used $|\cdot|$ to denote the desired extended geometric realization functor. We had already written $|\Delta^n|$ for the standard topological n-simplex. It will turn out that $|\cdot|$ applied to Δ^n gives the standard topological n-simplex, so there is no clash of notation.

5.1 Simplicial sets as colimits of Δ^n

By Theorem 4.3, any simplicial set S is a colimit of its tautological diagram $D_S : \mathcal{I}_S \to \mathbf{Set}_\Delta$. Explicitly,

$$\mathrm{Ob}(\mathcal{I}_S) = \coprod_{n \geq 0} \mathrm{Nat}(\Delta^n, S)$$

is the set of all the abstract simplices in S. The hom-sets of $\mathcal{I}_S$ are

$$\mathrm{Hom}_{\mathcal{I}_S}\big(\Delta^n \overset{\sigma}{\Rightarrow} S, \Delta^{n'} \overset{\sigma'}{\Rightarrow} S\big) = \Big\{\Delta^n \overset{a_*}{\Rightarrow} \Delta^{n'} \,:\, \sigma = \sigma' \circ a_*\Big\}.$$

Then S is a colimit of its tautological diagram $D_S : \mathcal{I}_S \to \mathbf{Set}_\Delta$, which takes

$$\big(\Delta^n \overset{\sigma}{\Rightarrow} S\big) \mapsto \Delta^n, \qquad \big(\Delta^n \overset{a_*}{\Rightarrow} \Delta^{n'}\big) \mapsto \big(\Delta^n \overset{a_*}{\Rightarrow} \Delta^{n'}\big).$$

(The accompanying morphisms $D_S(\sigma) = \Delta^n \to S$ are precisely the σ themselves.)

The interpretation is that S is made up of its abstract simplices $\Delta^n \overset{\sigma}{\Rightarrow} S$, with "gluing data" encoded in the arrows in $\mathcal{I}_S$.

5.2 Geometric realization

Eq. (5.1) says that when restricted to representables, we must have

$$\begin{aligned} |\cdot| : \Delta^n &\mapsto \mathrm{GR}([n]) = |\Delta^n|, \\ \big(\Delta^n \overset{a_*}{\Rightarrow} \Delta^{n'}\big) &\mapsto \mathrm{GR}\big([n] \overset{a}{\to} [n']\big) = \big(|\Delta^n| \overset{\tilde{a}}{\Rightarrow} |\Delta^{n'}|\big). \end{aligned}$$

Since a general simplicial set S has the form $S = \varinjlim D_S$, with D_S taking values in Δ^n's, we can compose D_S with the above $|\cdot|$ to obtain

$$|D_S| =: \mathcal{I}_S \to \mathbf{Top},$$

and then define

$$|S| := \varinjlim |D_S|. \tag{5.2}$$

as the *geometric realization* of S.

Explicitly, $|D_S|$ is the diagram

$$\begin{aligned} \big(\Delta^n \overset{\sigma}{\Rightarrow} S\big) &\mapsto |\Delta^n| \\ \big(\Delta^n \overset{a_*}{\Rightarrow} \Delta^{n'}\big) &\mapsto \big(|\Delta^n| \overset{\tilde{a}}{\to} |\Delta^{n'}|\big). \end{aligned}$$

So each "abstract gluing rule" a_* in S becomes "geometrically realized" as a concrete placement map $\tilde{a}$ of topological simplices. Taking the colimit of $|D_S|$ is the formal process of enforcing these placement maps as identification maps.

The last statement is made explicit by the following choice of colimits,

$$|S| \equiv \varinjlim |D_S| = \left(\coprod_{n \geq 0} \mathrm{Nat}(\Delta^n, S) \times |\Delta^n| \right) / \sim, \tag{5.3}$$

where $\sim$ is the equivalence relation generated by

$$\big(\sigma, \tilde{a}(t)\big) \sim \big(\sigma \circ a_*, t\big), \qquad \Delta^{n'} \overset{\sigma}{\Rightarrow} S,\ \Delta^n \overset{a_*}{\Rightarrow} \Delta^{n'}.$$

Here, the $\mathrm{Nat}(\Delta^n, S)$ are regarded as discrete sets, and the quotient topology is put on $|S|$. The accompanying morphisms are

$$|D_S|(\sigma) = |\Delta^n| \to |S| = \left(\coprod_{n \geq 0} \mathrm{Nat}(\Delta^n, S) \times |\Delta^n| \right) / \sim$$
$$t \mapsto [(\sigma, t)], \qquad \Delta^n \overset{\sigma}{\Rightarrow} \mathrm{Ob}(\mathcal{I}_S). \tag{5.4}$$

Remark. A technical property of **Top** allowing (5.2) to make sense is that it admits all small colimits. One first takes the colimit in **Set** (formula (5.3) is that from Example 4.1 at the level of sets), then puts the final topology on the colimit (the finest topology which makes all the accompanying maps to the colimit continuous).

5.3 Geometric realization as a functor

It is possible to deduce from the concrete formula (5.3) that geometric realization is a *functor*

$$|\cdot| : \mathbf{Set}_\Delta \to \mathbf{Top}.$$

If $\alpha : S \Rightarrow S'$, then the induced map $|S| \to |S'|$ is given by

$$[(\sigma, t)] \mapsto [(\alpha \circ \sigma, t)], \qquad \sigma \in \mathrm{Nat}(\Delta^n, S) \subseteq \mathrm{Ob}(\mathcal{I}_S),\ t \in |\Delta^n|.$$

More abstractly, if $\alpha : S \Rightarrow S'$, then there is an induced functor $A : \mathcal{I}_S \to \mathcal{I}_{S'}$, which on objects is given by

$$A : \left(\Delta^n \overset{\sigma}{\Rightarrow} S\right) \mapsto \left(\Delta^n \overset{\alpha \circ \sigma}{\Rightarrow} S'\right),$$

and on hom-sets is given by inclusions

$$A : \mathrm{Hom}_{\mathcal{I}_S}(\sigma, \sigma') \hookrightarrow \mathrm{Hom}_{\mathcal{I}_{S'}}(\alpha \circ \sigma, \alpha \circ \sigma').$$

Then

$$D_S = D_{S'} \circ A. \tag{5.5}$$

We have $|S'|$ being the vertex of a universal cone from the diagram $|D_{S'}|$. Then $|S'|$ is also the vertex of a cone from the diagram

$$|D_{S'}| \circ A = |D_S| : \mathcal{I}_S \to \mathbf{Top}.$$

By the universal property of colimits, we get a unique morphism

$$|S| \equiv \varinjlim |D_S| \to |S'|.$$

Functoriality of $|\cdot|$ follows[4] readily.

Theorem 5.1. *There is an adjunction* $|\cdot| \dashv \mathrm{Sing}$.

Proof. Let S be a simplicial set and X be a topological space. By the continuity result of Prop. 4.1, we have

$$\mathrm{Hom}_{\mathbf{Top}}(|S|, X) = \mathrm{Hom}_{\mathbf{Top}}(\varinjlim |D_S|, X) \cong \varprojlim \mathrm{Hom}_{\mathbf{Top}}(|D_S|(-), X).$$

For each object $\left(\Delta^n \overset{\sigma}{\Rightarrow} S\right) \in \mathcal{I}_S$, we have

$$D_S(\sigma) = \Delta^n, \qquad |D_S|(\sigma) = |\Delta^n|,$$

while in (3.7), we saw that the Yoneda Lemma implies

$$\mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X) \cong \mathrm{Hom}_{\mathbf{Set}_\Delta}(\Delta^n, \mathrm{Sing}(X)).$$

Thus

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Top}}(|S|, X) &\cong \varprojlim \mathrm{Hom}_{\mathbf{Top}}(|D_S|(-), X) && \text{(Continuity)} \\
&\cong \varprojlim \mathrm{Hom}_{\mathbf{Set}_\Delta}(D_S(-), \mathrm{Sing}(X)) && \text{(Yoneda)} \\
&\cong \mathrm{Hom}_{\mathbf{Set}_\Delta}(\varinjlim D_S, \mathrm{Sing}(X)) && \text{(Continuity)} \\
&= \mathrm{Hom}_{\mathbf{Set}_\Delta}(|S|, \mathrm{Sing}(X)).
\end{aligned}$$

Above, the bijections involving limits are natural in S (see (5.5)), while the Yoneda bijection is natural in $\mathrm{Sing}(X)$, thus in X (see Exercise 2.1). We conclude that the bijections

$$\mathrm{Hom}_{\mathbf{Top}}(|S|, X) \cong \mathrm{Hom}_{\mathbf{Set}_\Delta}(S, \mathrm{Sing}(X))$$

are natural in S and X. □

[4]More precisely, colimits are unique up to unique isomorphism. It we use a different choice of colimits when defining $|S|$, we will get a different functor $|\cdot|'$, naturally isomorphic to $|\cdot|$.

5.3.1 Extending geometric realization of standard simplicial sets

Let us adopt (5.3)–(5.4) as the colimits defining $|S|$. We wish to determine the geometric realization of $S = \Delta^n$. To avoid an a priori clash of notation, we write Δ^n_{top} for the standard topological n-simplex (instead of the usual $|\Delta^n|$.

For each $m \geq 0$, the Yoneda embedding gives

$$\text{Nat}(\Delta^n, \Delta^m) \overset{\text{Yoneda}}{\cong} \Delta^m([n]) = \text{Hom}_\Delta([n],[m]), \qquad n \geq 0.$$

So we may write the geometric realization of Δ^m as

$$|\Delta^m| = \left(\coprod_{n \geq 0} \text{Hom}_\Delta([n],[m]) \times |\Delta^n| \right) /_\sim.$$

Consider the map

$$\begin{aligned} \phi : \left(\coprod_{n \geq 0} \text{Hom}_\Delta([n],[m]) \times \Delta^n_{\text{top}} \right) &\to \Delta^m_{\text{top}} \\ (\sigma, t) &\mapsto \tilde{\sigma}(t). \end{aligned}$$

Here, $\tilde{\sigma} : \Delta^n_{\text{top}} \to \Delta^m_{\text{top}}$ is the (continuous) placement map corresponding to $\sigma \in \text{Hom}_\Delta([n],[m])$. Then ϕ is continuous, and it descends to a well-defined map on the quotient,

$$\begin{aligned} \varphi : |\Delta^m| &\to \Delta^m_{\text{top}} \\ [(\sigma, t)] &\mapsto \phi(\sigma, t) = \tilde{\sigma}(t), \end{aligned} \tag{5.6}$$

because if $\sigma = \sigma' \circ a$ for some $[n] \xrightarrow{a} [n']$, and $\sigma' \in \text{Hom}_\Delta([n'],[m])$, then

$$[(\sigma, t)] = [(\sigma' \circ a, t)] = [\sigma', \tilde{a}(t)] \overset{\varphi}{\mapsto} \tilde{\sigma'}(\tilde{a}(t)) = \widetilde{\sigma' \circ a}(t) = \tilde{\sigma}(t).$$

Also, φ is continuous by definition of quotient topology.

It is easily checked that the inverse of φ is the (continuous) map

$$\begin{aligned} \psi : \Delta^m_{\text{top}} &\to |\Delta^m| \\ t &\mapsto [(\text{id}_{[m]}, t)]. \end{aligned} \tag{5.7}$$

Therefore, $|\Delta^m|$ is homeomorphic to Δ^m_{top}, and it is justified to write $|\Delta^m|$ also for the latter.