15 Plus construction of higher K-theory (Not examinable)

15.1 Some topology

Almost no proofs of the topology results in this section will be provided, as there are many textbooks references for this, and the precise proofs are usually very lengthy (and not easy!). If you have never seen these homotopy theory and algebraic topology ideas, their use in the construction of K-theory may serve as some motivation and context.

CW complexes. When studying (basepoint-preserving) morphisms of (pointed) topological spaces "up to homotopy", it is sensible to formally promote those maps which induce isomorphisms on homotopy groups (and bijections on π_0) to become genuine isomorphisms. The latter kind of maps are called *weak homotopy equivalences*. The result of inverting these weak homotopy equivalences is the *homotopy category*.

It turns out that every topological space X admits a *CW-approximation*, in the sense that there is a weak homotopy equivalence $f : Z \to X$ where Z is a *CW-complex*. See, e.g., Hatcher's *Algebraic Topology* textbook, Chapter 4, *CW-approximation*. The upshot is that, up to equivalence of categories, the homotopy category can be defined as the category whose objects are CW-complexes, and whose morphisms are homotopy equivalence classes of continuous maps.

Recall that a CW complex is a topological space X constructed inductively from its n-skeleta X_n, $n \geq 0$.

- X_0 is a discrete set of points.
- Attach n-dimensional disks by gluing their boundary spheres S^{n-1} to X_{n-1} (quotient topology).
- $X = \cup_{n \geq 0} X_n = \varinjlim X_n$.

The attached disks are sometimes called the *cells* of X.

A CW complex should remind you of the geometric realization of a simplicial set. Indeed, after some tricky combinatorial manipulations, one may deduce that a geometric realization can be chosen to be a CW complex.

Fibrations. Some examples of fibrations $\pi : E \to B$ are:

- A cartesian product of topological spaces, $B \times F$ with projection map to B. The fibres $\pi^{-1}(x) = \{x\} \times F$ above various basepoints $x \in B$ are canonically homeomorphic to each other.
- A fibre bundle $\pi : E \to B$ with paracompact Hausdorff B. Fibres $\pi^{-1}(x)$ over various basepoints x in (the same connected component of) are non-canonically homeomorphic.

A fibration is a bit like a fibre bundle, but the focus is not on local trivializability (locally homeomorphic to product structure), but on lifting homotopies of B-valued maps to homotopies of E-valued maps. To be specific: a continuous map $\pi : E \to B$ has the *homotopy lifting property*, if given

- Any homotopy $h : X \times [0,1] \to B$,
- Any lift $H_0 : X \to E$ of the initial map $h_0 : X \to B$, (i.e., $\pi \circ H_0 = h_0$),

there is a homotopy $H : X \times [0,1] \to E$ lifting h (i.e., $\pi \circ H = h$). For example, take $X = \mathrm{pt}$, and think of a path in the base space being lifted to a path in the total space, with a prescribed starting point.

Definition 25. A *fibration* is a continuous map $\pi : E \to B$ satisfying the homotopy lifting property. The *fibre* at $x \in B$ is the subspace $\pi^{-1}(x)$.

It can be shown that the homotopy lifting property implies that for points in the same path-component of B, the fibres are homotopy equivalent (not necessarily homeomorphic). So we usually just say "the fibre" F, with the understanding that it is something to be studied up to homotopy equivalence. (In contrast, fibres of fibre bundles are homeomorphic.)

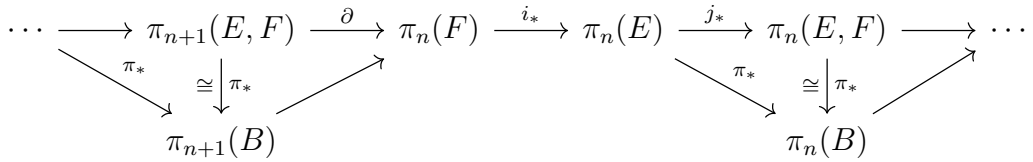
In general, for a pointed pair (X, A), meaning $A \subseteq X$ have a common basepoint x_0, there is a long exact sequence of homotopy groups,

$$\cdots \to \pi_{n+1}(X, A) \xrightarrow{\partial_{n+1}} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial_n} \cdots \xrightarrow{i_*} \pi_0(X).$$

The *relative homotopy group* $\pi_n(X, A)$ refers to the homotopy classes of pointed maps $D^n \to X$ such that $\partial D^n = S^{n-1}$ is mapped to A. (If A is the singleton set containing the basepoint, we get back $\pi_n(X)$.) Also j is the map which regards a pointed map $f : S^n \to X$ as a map $D^n \to X$ which restricts on

$\partial D^n = S^{n-1}$ to the constant map into A landing on the basepoint x_0. The connecting map ∂_n implements restriction-to-S^{n-1}.

In the case of a fibration $\pi : E \to B$, the inclusion of a fibre (above a chosen basepoint) gives the pointed pair (F, E), and then a long exact sequence as described above. This long exact sequence is the horizontal sequence in the diagram below,



One further argues that $\pi_* : \pi_n(E, F) \to \pi_n(B)$ is an isomorphism. Then the zig-zag diagram above gives the long exact sequence associated to the fibration $\pi : E \to B$,

$$\cdots \longrightarrow \pi_{n+1}(B) \xrightarrow{\partial \circ \pi_*^{-1}} \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{\pi_*} \pi_n(B) \longrightarrow \cdots \tag{15.1}$$

Classifying spaces. As a typical application of the exact sequence (15.1):

- If $\pi : E \to B$ is a covering map, thus F is a discrete set, then

$$\pi_n(E) \cong \pi_n(B), \qquad n \geq 2. \tag{15.2}$$

  If E, B are path-connected, then there is a bijection of sets

$$\pi_1(B) \cong \pi_1(E) \times \underbrace{\pi_0(F)}_{F}.$$

  Note that $\pi_* : \pi_1(E) \hookrightarrow \pi_1(B)$ is an injective group homomorphism. We say that E is a *universal cover* if $\pi_1(E) = 0$.

- Given a *topological* group G, there exists a weakly contractible[29] space EG, equipped with a free and proper G-action, and the quotient space $BG = EG/G$ is called a *classifying space*[30]. The sequence (15.1) simplifies because EG has trivial π_n's.

[29] Meaning all its homotopy groups are trivial.

[30] The geometric meaning is that any principal G-bundle over a (paracompact) space X is obtained by pulling back $\pi : EG \to BG$ via some "classifying map" $f : X \to BG$.

For a group G with *discrete* topology, the map $EG \to BG$ is a covering map with fibres G, and $\pi_n(G) = 0$ for all $n \geq 1$. Then (15.1) leads to $\pi_1(BG) \cong G$ and $\pi_n(BG) = 0$ for other n. A space with a single non-trivial homotopy group G is called an *Eilenberg–Maclane space*, and one often writes $BG = K(G, 1)$ (discrete topology case).

Remark. Often a group G arises with a natural *non-discrete* topology, e.g., Lie groups. For the Lie group $\mathrm{U}(1)$, a classifying space $B\mathrm{U}(1)$ is $\mathbb{CP}^\infty$ (the infinite complex projective space), which is simply-connected. In this case, $B\mathrm{U}(1)$ is extremely far from being an Eilenberg–Maclane space. Unless otherwise stated, we are always dealing with *discrete* groups.

For discrete G, it is possible to construct EG explicitly as a CW-complex, equipped with a free G-action which maps cells to cells. By a theorem of Whitehead, such a CW-complex realization of EG is actually contractible[31] (not just weakly). Furthermore, the quotient space $BG = EG/G$ is again CW-complex. One method is called the "infinite join construction", and we describe it with some examples.

Example 15.1. Let $G = \mathrm{pt}$ be the one-element group. Start with $X_0 = \mathrm{pt}$. Then inductively take X_n to be the cone[32] on X_{n-1}. So $X_n = \Delta^n_{\mathrm{top}}$ is the topological n-simplex. Take EG to be the colimit of the CW-complex inclusions $X_{n-1} \hookrightarrow X_n$.

The above choice of EG as an "infinite simplex" is perhaps more complicated than necessary. Any contractible space (such as a single point) also works as an EG.

Example 15.2. Intuitively, given spaces X, Y inside some ambient Euclidean space, the *join* $X * Y$ is the union of all line segments joining X to Y. For example, $X * S^0$ comprises two cones over X, with cone bases identified. This is the *suspension* of X.

Let $G = \mathbb{Z}_2$, identified as a space with S^0. Then EG is the infinite join

$$EG = G * G * G * \cdots .$$

In more detail: Start with $X_0 = G = S^0$. Then $X_1 = X_0 * G = X_0 * S^0$ is the suspension of S^0, which is topologically a circle S^1 (having also a certain

[31] Meaning that the identity map on EG is homotopic to a constant map.

[32] Take $X_{n-1} \times [0, 1]$ and collapse $X_{n-1} \times \{1\}$ to a single point.

CW-complex structure). Next, X_2 is the suspension of a circle, which is S^2. Inductively, we arrive at the "infinite sphere" $EG = S^\infty$.

Initially, G acts on X_0 by swapping the two points. Then on X_1, G swaps the cross-sectional S^0 and the two cones in the suspension — this is the antipodal map on S^1. Inductively, the G-action on $X_n = S^n$ is the antipodal map. Modding out $EG = S^\infty$ by the antipodal map gives the "infinite real projective space" $\mathbb{RP}^\infty$. Let us also mention that, intuitively, in the join construction, adding $(n+1)$-cells to $X_n = S^n$ kills its n-th homotopy group, and in the colimit, no homotopy group survives.

Remark. If $\phi : G \to H$ is a group homomorphism, we obtain an induced map on the n-fold joins,

$$G * G * \cdots * G \to H * H * \cdots * H.$$

Check that there is a well-defined map $(G*G*\cdots*G)/G \to (H*H*\cdots*H)/H$, and take the $n \to \infty$ limit, to get $B\phi : BG \to BH$. This means that the classifying space construction is a functor from the category of groups to the homotopy category.

Remark. We can also regard G as a small category, and consider the nerve NG, and take the geometric realization $|NG|$ to get another CW-complex model of BG (discrete topology on G). See Example 8.1.7 of Weibel's *An Introduction to Homological Algebra*.

Hurewicz Theorem. For a topological space X, we would usually use the singular chain complex to define the (singular) homology groups of X, denoted

$$H_n(X) \equiv H_n(X; \mathbb{Z}).$$

It is a standard result that

$$H_n(S^n) \cong \mathbb{Z}.$$

A generator can be described as follows. Identify $\Delta^{n+1}_{\text{top}}$ with a disc D^{n+1}, and the boundary of $\Delta^{n+1}_{\text{top}}$ with $\partial D^{n+1} = S^n$. The identity map on $\Delta^{n+1}_{\text{top}}$ becomes viewed as a singular $(n+1)$-simplex $\Delta^{n+1}_{\text{top}} \to D^{n+1}$, whose boundary is a singular n-chain on S^n (a formal alternating sum of maps $\Delta^n_{\text{top}} \to S^n$). The latter singular n-chain is a cycle, and its homology class is a generator of $H_n(S^n)$. This generator is typically written

$$[S^n],$$

and is called a "fundamental class" for S^n.

If X is a pointed space, there are *Hurewicz homomorphisms*

$$\begin{aligned} \pi_n(X) &\to H_n(X) \\ \big(f : S^n \to X\big) &\mapsto f_*[S^n]. \end{aligned}$$

The Hurewicz theorems say the following.

- When X is path-connected, the Hurewicz homomorphism descends to an isomorphism
$$\pi_1(X)_{\mathrm{ab}} \to H_1(X).$$
- Let $n \geq 2$. If $\pi_j(X) = 0$ for all $j < n$, then the homology groups $H_j(X)$ vanish for $1 \leq j < n$ as well. Furthermore, the Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ is an isomorphism.

15.2 Relating homology of BG and homology of G

(See algebraic topology textbooks for detailed constructions.) For a CW-complex X, it is convenient that we can use the cellular chain complex instead of the singular chain complex. The *cellular chain groups* $C_n(X)$ are the free abelian groups on the set of n-cells in X. The maps attaching the n-cells to the $(n-1)$-skeleton can be combined into a homomorphism $C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$, which assemble to give the *cellular chain complex* of X. The homology groups of this chain complex are the *(cellular) homology groups* of X,

$$H_n(X) = \ker \partial_n / \mathrm{Im}\, \partial_{n+1}.$$

Let X be path-connected and $G = \pi_1(X)$. There is a universal cover $\pi : \widetilde{X} \to X$, with $\pi_1(\widetilde{X}) = 0$. In fact, $\widetilde{X}$ has a CW-complex structure, and a free action of $G = \pi_1(X)$ with quotient X. The cellular chain groups $C_n(\widetilde{X})$ are upgraded to free $\mathbb{Z}[G]$-modules. If M is a $\mathbb{Z}[G]$-module, we can take the twisted chain groups $C_n(\widetilde{X}) \otimes_{\mathbb{Z}[G]} M$ with twisted boundary maps $\partial_n \otimes \mathrm{id}_M$, then the resulting homology groups are the *(cellular) homology groups of X with local coefficients M*. If M is the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$, we have $C_n(\widetilde{X}) \otimes_{\mathbb{Z}[G]} M \cong C_n(X)$, and recover the situation of ordinary coefficients $\mathbb{Z}$.

We are concerned with the case $X = BG$, which has $\pi_1(BG) = G$ and EG as universal cover.

Theorem 15.1. *For any group*[33] *G, there is a natural isomorphism and $\mathbb{Z}[G]$-module M, there is a natural isomorphism*

$$H_n(G; M) \cong H_n(BG; M)$$

between the homology of the group G with coefficients M, and the homology of the space BG with local coefficients M.

Proof.

- EG has a free G action, so its chain complex $C_\bullet(EG)$ has the form
$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to 0,$$
with F_i being free $\mathbb{Z}[G]$-modules. But EG is also contractible, so general theory (homotopy invariance of homology) says that the above is exact (except in degree 0), and we get a free resolution
$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0.$$

- For the homology of G, we used another free resolution (14.9) of $\mathbb{Z}$. Appealing to general theory from homological algebra, applying $(-)\otimes_{\mathbb{Z}[G]} M$ to either resolution leads to chain complexes with the same homologies. We refer to pp. 263 of Rosenberg's book, Chapter IV.3 of Weibel's book, and references therein, for more details of this black box.

□

For unital rings R, we know that

$$\begin{aligned}
K_1(R) &= \mathrm{GL}(R)_{\mathrm{ab}} \\
&= H_1(\mathrm{GL}(R); \mathbb{Z}) && \text{(Example 14.1)} \\
&\cong H_1(B\mathrm{GL}(R); \mathbb{Z}) && \text{(Theorem 15.1)} \\
&\cong \pi_1(B\mathrm{GL}(R))_{\mathrm{ab}} && \text{(Hurewicz)} \\
&\cong \pi_1(B\mathrm{GL}(R))/\mathrm{E}(R). && (\mathrm{E}(R) = [\mathrm{GL}(R), \mathrm{GL}(R)])
\end{aligned}$$

We also have

$$\begin{aligned}
\pi_2(B\mathrm{E}(R); \mathbb{Z}) &\overset{\text{Hurewicz}}{\to} H_2(B\mathrm{E}(R); \mathbb{Z}) \\
&\cong H_2(\mathrm{E}(R); \mathbb{Z}) && \text{(Theorem 15.1)} \\
&\cong K_2(R). && \text{(Corollary 14.7)}
\end{aligned}$$

[33] With discrete topology!

Quillen had the idea that $K_n(R)$ should be π_n of some appropriate space, and the above equations show that $B\mathrm{GL}(R)$ "almost works" for K_1, while $B\mathrm{E}(R)$ "almost works" for K_2.

We know that $\mathrm{E}(R)$ is a perfect normal subgroup of $\mathrm{GL}(R)$. What is needed is a functorial way to modify

$$\begin{aligned} B\mathrm{GL}(R) &\rightsquigarrow B\mathrm{GL}(R)^+ \\ \mathrm{E}(R) &\rightsquigarrow \mathrm{E}(R)^+, \end{aligned}$$

such that the homotopy groups of the plus-spaces get modified to

$$\begin{aligned} &\pi_1(B\mathrm{GL}(R)^+) \cong \pi_1(B\mathrm{GL}(R))/\mathrm{E}(R), \\ &\pi_2(B\mathrm{GL}(R)^+) \cong \pi_2(B\mathrm{E}(R)^+) \cong H_2(B\mathrm{E}(R); \mathbb{Z}). \end{aligned} \tag{15.3}$$

Then

$$K_n(R) \cong \pi_n(B\mathrm{GL}(R)^+), \qquad n = 1, 2,$$

would hold. It would then be reasonable to *define* $K_n(R)$ by $\pi_n(B\mathrm{GL}(R)^+)$ for $n > 2$, as a generalization of the classical K_1, K_2.

The solution to (15.3) was found by Quillen, using the so-called *plus construction* of Kervaire, applied to the space $B\mathrm{GL}(R)$.

15.3 Plus construction

Let X be a space with universal cover of $\widetilde{X} \to X$. The fundamental group $G = \pi_1(X)$ acts freely on $\widetilde{X}$ by deck transformations. Let π be a normal subgroup G. On the quotient space $\widetilde{X}/\pi$, there remains a free action of the quotient group G/π. Modding out by this (G/π)-action determines a covering space $p : \widetilde{X}/\pi \to X$, with fibres G/π. By the long exact sequence, $p_* : \pi_1(\widetilde{X}/\pi) \to \pi_1(X)$ is precisely the inclusion $\pi \hookrightarrow \pi_1(X)$.

Let X be a connected CW-complex, and fix a *perfect* normal subgroup

$$\pi \subseteq G = \pi_1(X)$$

of its fundamental group. The *plus construction* (with respect to π) is a certain functorial construction of a new space X^+ such that π is eliminated from the fundamental group. Specifically, X^+ is obtained from X by attaching certain 2-cells and 3-cells, such that

$$\pi_1(X^+) = G/\pi,$$

and furthermore, the inclusion $\iota : X \to X^+$ induces

- The quotient map $G \to G/\pi$ on the fundamental groups.
- Isomorphisms $H_n(X; M) \to H_n(X^+; M)$ on the homology groups, for any $\mathbb{Z}[G/\pi]$-module M as local coefficients.

The plus construction is sketched at the end of this subsection. Granted its availability, let us apply it to $X = BG$ with universal cover $\widetilde{X} = EG$. The given normal subgroup $\pi \subseteq G$ acts freely on EG, so that $EG/\pi = B\pi$. There is a residual action of G/π, i.e.,

$$p : B\pi \to BG$$

is a covering map with fibres G/π. When we apply the plus construction to BG to get BG^+, we can lift the construction to $B\pi$ to get a covering $p^+ : B\pi^+ \to BG^+$ (with fibres G/π). Then recalling (15.2), we have

$$\pi_j(B\pi^+) \cong \pi_j(BG^+), \qquad j \geq 2. \tag{15.4}$$

The notation $B\pi^+$ is meant to indicate that $B\pi^+$ described above indeed gives a plus construction for $B\pi$, with π trivially regarded as a perfect normal subgroup of $\pi_1(B\pi) = \pi$. Specifically, the homology-preservation property is inherited from that of BG^+.

Note that $\pi_1(B\pi^+) = \pi_1(B\pi)/\pi = \pi/\pi = 0$, i.e., $B\pi^+$ is now *simply-connected*. So we may apply the Hurewicz theorem, in combination with (15.4), to get

$$\pi_2(BG^+) \cong \pi_2(B\pi^+) \cong H_2(B\pi^+; \mathbb{Z}).$$

In particular, applying the above discussion to $G = \mathrm{GL}(R)$, we have found that the space $B\mathrm{GL}(R)^+$ fulfils the requirements (15.3).

Definition 26. Let R be a unital ring. Then the higher K-groups are defined as

$$K_n(R) = \pi_n(B\mathrm{GL}(R)^+ \times K_0(R)), \qquad n \geq 0.$$

The extra $K_0(R)$ factors are put in by hand, to ensure that the $n = 0$ case recovers the Grothendieck group $K_0(R)$. Admittedly, this is somewhat ad-hoc.

15.3.1 Sketch of plus construction.

Let Λ be a generating set for $\pi \subseteq G = \pi_1(X)$. So each $i \in \Lambda$ determines a homotopy class of a loop $g_i : S^1 \to X$. Since these i are commutators (because $\pi = [\pi, \pi]$), they are killed by the Hurewicz map to $H_1(X)$.

For each $i \in \Lambda$, we attach a 2-cell to X, gluing the boundary circle to X according to the loop g_i. Call the resulting space Y. The loops g_i in X now become contractible in Y. So the inclusion $\iota : X \to Y$ induces the quotient map

$$\iota : \underbrace{\pi_1(X)}_{G} \to \underbrace{\pi_1(Y)}_{G/\pi}.$$

We have killed π in the fundamental group, but the homology has been disturbed.

Let $\widetilde{Y} \to Y$ be the universal cover, with fibre G/π. Pullback to a covering space $\widetilde{X} \to X$, also with fibre G/π. We have $\pi_1(\widetilde{X}) = \pi$, and so $H_1(\widetilde{X}) = \pi_{\text{ab}} = 0$ (as π is perfect). Note that $\widetilde{Y}$ is obtained from $\widetilde{X}$ by attaching 2-cells e_i^2, in the same way as Y from X. So the relative homology $H_2(Y, X; \mathbb{Z})$ is free abelian on the extra 2-cells e_i^2, while $H_2(\widetilde{Y}, \widetilde{X}; \mathbb{Z})$ is the free $\mathbb{Z}[G/\pi]$-module on the extra e_i^2. The relative homologies are trivial in other degrees. Since $H_1(\widetilde{X}) = 0$, the homology long exact sequence shows that $H_2(\widetilde{Y}; \mathbb{Z})$ is $H_2(\widetilde{X}; \mathbb{Z})$ direct summed with the free $\mathbb{Z}[G/\pi]$-module on the extra 2-cells e_i^2.

We want to kill off the extra homology summands introduced into $\widetilde{Y}$ and Y. So pick maps $h_i : S^2 \to Y$ which map to $[e_i^2]$ under the Hurewicz map (these exist because $\pi_1(\widetilde{Y}) = 0$.) Then attach 3-cells e_i^3 to Y by gluing their boundary S^2 according to the maps h_i, and call the resulting space X^+. The idea is that in X^+, we have $e_i^2 = \partial e_i^3$, so the e_i^2 no longer contributes extra homology groups. This generalizes to the twisted boundary maps in the case of local coefficients.

There are some further technical properties of the plus construction:

- Up to homotopy (relative to X), X^+ is the unique CW-complex containing X and satisfying the two defining conditions.
- It is functorial with respect to the data $(X, \pi \subseteq \pi_1(X))$.

15.4 Discussion

15.4.1 K_3-group.

We can say a little more about $K_3(R) \equiv \pi_3(B\mathrm{GL}(R)^+)$. First, (15.4) gives

$$K_3(R) \equiv \pi_3(BG^+) \cong \pi_3(B\pi^+) \tag{15.5}$$

for $G = \mathrm{GL}(R)$, $\pi = \mathrm{E}(R)$. To understand $\pi_3(B\pi^+)$ better, we will relate it to the universal central extension (E, φ) of π (which is the Steinberg group).

Applying the classifying space functor to $1 \to H_2(\pi;\mathbb{Z}) \to E \xrightarrow{\varphi} \pi \to 1$ (the exact sequence from Corollary 14.7), we get a fibration $BE \xrightarrow{B\varphi} B\pi$, with fibre $BH_2(\pi,\mathbb{Z})$. Applying the functorial plus construction, we get $BE^+ \xrightarrow{B\varphi^+} B\pi^+$. Up to homotopy, we may assume that $B\varphi^+$ is a fibration, and we write F for the (homotopy) fibre. Concisely, we have the diagram

$$\begin{array}{ccccc} BH_2(\pi;\mathbb{Z}) & \longrightarrow & BE & \xrightarrow{B\varphi} & B\pi \\ & & \downarrow & & \downarrow \\ F & \longrightarrow & BE^+ & \xrightarrow{B\varphi^+} & B\pi^+ \end{array} . \tag{15.6}$$

We will compute

$$\pi_3(BE^+) \cong H_3(E;\mathbb{Z}), \tag{15.7}$$

then argue that

$$\pi_3(BE^+) \overset{B\varphi^+_*}{\cong} \pi_3(B\pi^+) \overset{(15.5)}{\cong} K_3(R). \tag{15.8}$$

By Theorem 14.1, E is a perfect group, yet it does not admit non-trivial central extensions, so it must be its own universal central extension, with trivial kernel. This trivial kernel equals $H_2(E;\mathbb{Z})$ because of Theorem 14.6. So

$$H_2(BE^+) \cong H_2(BE) \overset{\text{Thm. 15.1}}{\cong} H_2(E;\mathbb{Z}) = 0.$$

Now BE^+ is simply-connected since the plus construction kills the E part of the fundamental group $\pi_1(BE) = E$. Then the Hurewicz theorem gives an isomorphism $\pi_2(BE^+) \cong H_2(BE^+) = 0$. By the Hurewicz theorem again,

$$\pi_3(BE^+) \cong H_3(BE^+;\mathbb{Z}) \cong H_3(E;\mathbb{Z}),$$

which is the claim (15.7).

Finally, we need a difficult argument that the fibre F in (15.6) can be taken to be $BH_2(\pi;\mathbb{Z})$; see pp. 271–273 and Remark 5.2.9 of Rosenberg's book. Granted this, the long exact sequence for $F \to BE^+ \overset{B\varphi^+}{\to} B\pi^+$ will give $\pi_j(BE^+) \overset{B\varphi^+_*}{\cong} \pi_j(B\pi^+), j \geq 3$. The $j = 3$ case is the claim (15.8).

To conclude, the results (15.7)–(15.8) and $E = \mathrm{St}(R)$ say that

$$K_3(R) \cong H_3(\mathrm{St}(R);\mathbb{Z}).$$

15.4.2 Relative K-groups.

(Sketch.) If I is an ideal in a unital ring R, with quotient map $q : R \to R/I$, we write

$$\mathbf{K}(R) = B\mathrm{GL}(R)^+ \times K_0(R), \qquad \mathbf{K}(R/I) = B\mathrm{GL}(R/I)^+ \times K_0(R/I).$$

Their homotopy groups give the absolute K_n groups for R and R/I. Functoriality follows from that of B and the plus construction.

We have the map

$$\mathbf{K}(R) \overset{Bq^+}{\longrightarrow} \mathbf{K}(R/I).$$

understood to be in the homotopy category. We write the homotopy fibre of Bq^+ as $\mathbf{K}(R, I)$, and its homotopy groups are the relative groups $K_n(R, I)$. So there is a fibration

$$\mathbf{K}(R, I) \longrightarrow \mathbf{K}(R) \longrightarrow \mathbf{K}(R/I),$$

whose long exact sequence of homotopy groups is (by construction/definition) the long exact sequence of K-theory groups.