10 Relative K_1

Let us abstract the *algebraic* properties of the Fredholm determininant homomorphism, Theorem 9.7, to motivate relative K_1-theory.

In what follows, R is a (noncommutative) unital ring, and I is an ideal in R.

- The quotient map $R \to R/I$ restricts to a group homomorphism $R^\times \to (R/I)^\times$, whose kernel is denoted

$$\mathrm{GL}_1(R, I) := \{U \in R^\times \,:\, U = 1 + a,\, a \in I\}.$$

- Passing to matrix algebras, for each $n \in \mathbb{N}$, we also have

$$\mathrm{GL}_n(R, I) := \ker\big(\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/I)\big) = (1_n + M_n(I))^\times,$$

  which is a normal subgroup of $\mathrm{GL}_n(R)$. There are the usual upper-left embeddings

$$\mathrm{GL}_m(R, I) \hookrightarrow \mathrm{GL}_n(R, I), \qquad U \mapsto \mathrm{diag}(U, 1_{n-m}), \qquad n \geq m.$$

  The colimit is denoted

$$\mathrm{GL}_\infty(R, I) = (1_\infty + M_\infty(I))^\times,$$

  and it is a normal subgroup of $\mathrm{GL}_\infty(R)$.

For the motivating example

$$R = \mathcal{L}(\mathcal{H}), \qquad I = \mathcal{L}^1(\mathcal{H}), \tag{10.1}$$

$\mathrm{GL}_1(\mathcal{L}(\mathcal{H}), \mathcal{L}^1(\mathcal{H}))$ is precisely the group of invertible determinant class operators on $\mathcal{H}$. Moreover, $M_n(\mathcal{L}(\mathcal{H}))$ is identified with $\mathcal{L}(\mathcal{H}^{\oplus n})$. Thus $\mathrm{GL}_n(\mathcal{L}(\mathcal{H}), \mathcal{L}^1(\mathcal{H}))$ is the group of invertible determinant class operators on the direct sum Hilbert space $\mathcal{H}^{\oplus n}$. So there are Fredholm determinant homomorphisms

$$\det{}^{(n)} : \mathrm{GL}_n(\mathcal{L}(\mathcal{H}), \mathcal{L}^1(\mathcal{H})) \to \mathbb{C}^\times, \qquad n \in \mathbb{N}.$$

Because

$$\det{}^{(n)} \begin{pmatrix} U & 0 \\ 0 & 1_{n-m} \end{pmatrix} = \det{}^{(m)} U, \qquad U \in \mathrm{GL}_m(\mathcal{L}(\mathcal{H}), \mathcal{L}^1(\mathcal{H})),$$

these $\det^{(n)}$ are compatible with the embeddings $\mathrm{GL}_m(\mathcal{L}(\mathcal{H}), \mathcal{L}^1(\mathcal{H})) \hookrightarrow \mathrm{GL}_n(\mathcal{L}(\mathcal{H}), \mathcal{L}^1(\mathcal{H}))$. Thus there is a well-defined Fredholm determinant homomorphism

$$\det : \mathrm{GL}_\infty(\mathcal{L}(\mathcal{H}), \mathcal{L}^1(\mathcal{H})) \to \mathbb{C}^\times. \tag{10.2}$$

Furthermore, $\det^{(n)} : \mathrm{GL}_n(\mathcal{L}(\mathcal{H}), \mathcal{L}^1(\mathcal{H})) \to \mathbb{C}^\times$ is invariant under conjugation by $\mathrm{GL}_n(\mathcal{L}(\mathcal{H}))$. Likewise, $\det : \mathrm{GL}_\infty(\mathcal{L}(\mathcal{H}), \mathcal{L}^1(\mathcal{H})) \to \mathbb{C}^\times$ is invariant under conjugation by $\mathrm{GL}_\infty(\mathcal{L}(\mathcal{H}))$.

Let us investigate the kernel of det. An $n \times n$ elementary matrix $e_{ij}(T)$ with $T \in \mathcal{L}^1(\mathcal{H})$ belongs to $\mathrm{GL}_n(\mathcal{L}(\mathcal{H}), \mathcal{L}^1(\mathcal{H}))$, since it has the form $1_{\mathcal{H}^{\oplus n}} + T$ for some trace class $\tilde{T} \in \mathcal{L}^1(\mathcal{H}^{\otimes n})$. So its Fredholm determinant is defined, and more importantly, its determinant is trivial. To see this, it suffices to consider $n = 2$, and

$$e_{12}(T) = \begin{pmatrix} 1_{\mathcal{H}} & T \\ 0 & 1_{\mathcal{H}} \end{pmatrix}.$$

Any finite-rank approximation F_n to T also gives $\begin{pmatrix} 0 & F_n \\ 0 & 0 \end{pmatrix}$ approximating $\tilde{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$. Let $\mathcal{H} = V \oplus W$ be a decomposition of $\mathcal{H}$ satisfying the usual condition (9.11). With respect to $\mathcal{H}^{\oplus 2} = V \oplus W \oplus V \oplus W$, we have

$$\begin{pmatrix} 1_{\mathcal{H}} & F_n \\ 0 & 1_{\mathcal{H}} \end{pmatrix} = \begin{pmatrix} 1_V & 0 & F_n|_V & 0 \\ 0 & 1_W & 0 & 0 \\ 0 & 0 & 1_V & 0 \\ 0 & 0 & 0 & 1_W \end{pmatrix} \cong \begin{pmatrix} 1_V & F_n|_V & 0 \\ 0 & 1_V & 0 \\ 0 & 0 & 1_{W \oplus W} \end{pmatrix},$$

so

$$\det \begin{pmatrix} 1_{\mathcal{H}} & F_n \\ 0 & 1_{\mathcal{H}} \end{pmatrix} = \det_{V \oplus V} \begin{pmatrix} 1_V & F_n|_V \\ 0 & 1_V \end{pmatrix} = 1.$$

Then

$$\det(e_{12}(T)) = \lim_{n \to \infty} \det \begin{pmatrix} 1_{\mathcal{H}} & F_n \\ 0 & 1_{\mathcal{H}} \end{pmatrix} = 1.$$

By conjugation invariance, conjugates of $e_{ij}(T), T \in \mathcal{L}^1(\mathcal{H})$ by $\mathrm{GL}_\infty(\mathcal{L}(\mathcal{H}))$ elements also have trivial determinant. Therefore, the normal closure

$$\mathrm{ncl}_{\mathrm{GL}_\infty(R)}\{e_{ij}(r) \,:\, r \in I\} \subseteq \mathrm{GL}_\infty(R, I)$$

lies in $\ker \det$. Explicitly, this normal closure comprises all possible products of

$$Se_{ij}(r)S^{-1}, \qquad r \in I,\ S \in \mathrm{GL}_\infty(R),\ i \neq j \in \mathbb{N}.$$

Thus, det actually descends to the quotient of $\mathrm{GL}_\infty(R)$ by this normal closure.

10.1 Relative K_1-group

In the general setting of an ideal I in a unital ring R, suppose we have a surjective group homomorphism

$$D : \mathrm{GL}_\infty(R, I) \to G$$

with the following two properties:

- D is trivial on all elementary matrices $e_{ij}(r)$ with $r \in I$;
- D is invariant under conjugation by $\mathrm{GL}_\infty(R)$.

We shall call such a homomorphism a *relative determinant homomorphism*, and $\mathrm{GL}_\infty(R, I) \subseteq \mathrm{GL}_\infty(R)$ the (normal) subgroup of *D-class* invertibles.

Below, we will see that a relative determinant homomorphism D descends to a homomorphism on a certain abelian quotient of $\mathrm{GL}_\infty(R, I)$, denoted $K_1(R, I)$. In other words, we get a surjective homomorphism of abelian groups,

$$\check{D} : K_1(R, I) \to G.$$

(A posteriori, G must be abelian as well.)

The homomorphism $\check{D}$ should be compared to the "absolute determinants" $\check{\det} : K_1(R) \to R^\times_{\mathrm{ab}}$ (defined for all commutative rings R and certain noncommutative ones), studied in Section 7.3.

Definition 11. Let I be an ideal in a unital ring R. Set

$$\mathrm{E}_\infty(R, I) := \mathrm{ncl}_{\mathrm{E}_\infty(R)}\{e_{ij}(r) \,:\, r \in I\} \tag{10.3}$$
$$= \mathrm{ncl}_{\mathrm{GL}_\infty(R)}\{e_{ij}(r) \,:\, r \in I\}, \tag{10.4}$$

and define the *relative K_1-group* to be the quotient

$$K_1(R, I) := \mathrm{GL}_\infty(R, I)/\mathrm{E}_\infty(R, I). \tag{10.5}$$

Some remarks on Definition 11:

- A priori, (10.4) is larger than (10.3), since more conjugates are being admitted. However, for any conjugate

$$Se_{ij}(r)S^{-1}, \qquad S \in \mathrm{GL}_n(R),\ r \in I,$$

we have the factorization

$$\begin{pmatrix} Se_{ij}(r)S^{-1} & 0 \\ 0 & 1_n \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} e_{ij}(r) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & S \end{pmatrix}.$$

The Whitehead Lemma 7.3 says that $\begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix} \in \mathrm{E}_{2n}(R)$, so the above factorization actually shows that $Se_{ij}(r)S^{-1}$ can be regarded as a conjugate of $e_{ij}(r)$ by some $\mathrm{E}_\infty(R)$ element. This shows that (10.4) and (10.3) are equivalent.

- Since each $e_{ij}(r), r \in I$, lies in $\mathrm{GL}_\infty(R, I)$, all its conjugates $Se_{ij}(r)S^{-1}$ by $S \in \mathrm{GL}_\infty(R)$ also lie in $\mathrm{GL}_\infty(R, I)$. Thus $\mathrm{E}_\infty(R, I)$ is a (normal) subgroup of $\mathrm{GL}_\infty(R, I)$ (not just of $\mathrm{GL}_\infty(R)$), and the quotient group (10.5) makes sense.

  The more "efficient" definition (10.3) of $\mathrm{E}_\infty(R, I)$ shows that $\mathrm{E}_\infty(R, I)$ is a subgroup of $\mathrm{E}_\infty(R)$ (not just a subgroup of $\mathrm{GL}_\infty(R)$). So we have a commuting diagram of short exact sequences,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{E}_\infty(R, I) & \longrightarrow & \mathrm{GL}_\infty(R, I) & \xrightarrow{\text{quotient}} & K_1(R, I) & \longrightarrow & 1 \\ & & \downarrow\subseteq & & \downarrow\subseteq & & \vdots j & & \\ 1 & \longrightarrow & \mathrm{E}_\infty(R) & \longrightarrow & \mathrm{GL}_\infty(R) & \xrightarrow{\text{quotient}} & K_1(R) & \longrightarrow & 1 \end{array},$$

  inducing K-theory homomorphism

$$\begin{aligned} j : K_1(R, I) &\to K_1(R) \\ U \bmod \mathrm{E}_\infty(R, I) &\mapsto U \bmod \mathrm{E}_\infty(R). \end{aligned} \tag{10.6}$$

Lemma 10.1 (Relative Whitehead Lemma)**.** *Let I be an ideal in a unital ring R. Then $\mathrm{E}_\infty(R, I)$ contains the commutator subgroup of $\mathrm{GL}_\infty(R, I)$, thus $K_1(R, I)$ is abelian.*

Proof. Let $U, V \in \mathrm{GL}_\infty(R, I)$, regarded as $n \times n$ matrices for some n, and consider their multiplicative commutator taken in $2n \times 2n$ matrices,

$$\begin{pmatrix} UVU^{-1}V^{-1} & 0 \\ 0 & 1_n \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V^{-1} \end{pmatrix} \begin{pmatrix} (VU)^{-1} & 0 \\ 0 & VU \end{pmatrix}. \tag{10.7}$$

As in the Whitehead Lemma, each factor in (10.7) lies in $\mathrm{E}_{2n}(R)$, but we need the stronger statement that it lies in $\mathrm{E}_{2n}(R, I)$.

It will be enough to show that $\begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix} \in \mathrm{E}_{2n}(R, I)$ when $U \in \mathrm{GL}_n(R, I)$. Note that $U = 1_n + T$ for some $T \in M_n(I)$. Consider the factorization

$$\begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix} = \begin{pmatrix} 1_n & 1_n \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ T & 1_n \end{pmatrix} \begin{pmatrix} 1_n & -1_n \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_n & U^{-1}T \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ -UT & 1_n \end{pmatrix}. \tag{10.8}$$

The second factor in (10.8) is a lower triangular matrix with 1s along the diagonal, and off-diagonal entries in I. As in the proof of the Whitehead Lemma 7.3, it is a product of elementary matrices $e_{ij}(r)$ with $r \in I$. Therefore, the product of the first three factors lies in the normal closure $\mathrm{E}_{2n}(R, I)$. The last two triangular matrices in (10.8) are likewise in $\mathrm{E}_{2n}(R, I)$. Thus $\begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix} \in \mathrm{E}_{2n}(R, I)$. □

Proposition 10.2 (Half-exactness of K_1)**.** *Let I be an ideal in a unital ring R. Then there is an exact sequence of abelian groups,*

$$K_1(R, I) \to K_1(R) \to K_1(R/I).$$

Proof. In the above sequence, the first map $j : K_1(R, I) \to K_1(R)$ is given by (10.6). The second map $q_* : K_1(R) \to K_1(R/I)$ is induced by the quotient ring homomorphism $q : R \to R/I$.

- ($\operatorname{Im} j \subseteq \ker q_*$.) Any class in $K_1(R, I)$ has the form $[U]_{K_1(R,I)}$, where the representative U lies in $\mathrm{GL}_\infty(R, I) = \ker q$. Then

  $$q_* \circ j[U]_{K_1(R,I)} = q_*[U]_{K_1(R)} = [q(U)]_{K_1(R/I)} = 1,$$

  i.e., $j[U]_{K_1(R,I)} \in \ker q_*$.

- ($\ker q_* \subseteq \operatorname{Im} j$.) Let $[U]_{K_1(R)} \in \ker q_*$. So $U \in \mathrm{GL}_\infty(R)$ with $q(U) \in \mathrm{E}_\infty(R/I)$. Now, q also restricts to a surjective homomorphism $q : \mathrm{E}_\infty(R) \to \mathrm{E}_\infty(R/I)$. Therefore, $q(U) = q(E)$ for some $E \in \mathrm{E}_\infty(R)$. Then $UE^{-1} \in \mathrm{GL}_\infty(R/I)$ satisfies $q(UE^{-1}) = 1$, meaning that

  $$UE^{-1} \in \ker \big(q : \mathrm{GL}_\infty(R) \to \mathrm{GL}_\infty(R/I)\big) \equiv \mathrm{GL}_\infty(R, I).$$

  Thus the class $[UE^{-1}]_{K_1(R,I)}$ is defined, and by definition of j,

  $$j[UE^{-1}]_{K_1(R,I)} = [UE^{-1}]_{K_1(R)} = [U]_{K_1(R)}[E^{-1}]_{K_1(R)} = [U]_{K_1(R)}.$$

□

Example 10.1. The Fredholm determinant is a relative determinant homomorphism, so it descends to an abelian group homomorphism,

$$\check{\det} : K_1(\mathcal{L}(\mathcal{H}), \mathcal{L}^1(\mathcal{H})) \to \mathbb{C}^\times. \tag{10.9}$$

Furthermore, $\check{\det}$ admits a splitting: take any orthogonal splitting $\mathcal{H} = V \oplus V^\perp$ with one-dimensional V. Then for $\lambda \in \mathbb{C}^\times$, the operator

$$\lambda \oplus 1_{V^\perp}$$

is an element of $\mathrm{GL}_1(\mathcal{L}(\mathcal{H}), \mathcal{L}^1(\mathcal{H}))$ whose determinant is λ. Therefore,

$$K_1(\mathcal{L}(\mathcal{H}), \mathcal{L}^1(\mathcal{H})) \cong \ker \check{\det} \oplus \mathbb{C}^\times.$$

10.1.1 Failure of excision for $K_1(R, I)$

In an exercise, we found that the groups $\mathrm{GL}_n(R, I)$ are independent of the choice of unital ring R in which I is realized as an ideal, thus $\mathrm{GL}_\infty(R, I)$ depends only on I as a (possibly non-unital) ring. Suppose $I \subseteq R$ is a proper ideal. This suggests that we can "excise" most of R away, and just use the "miminal" unital ring I^+ containing I as an ideal. This ring I^+ has underlying abelian group being $\mathbb{Z} \oplus I$, and multiplication rule being

$$(m, r)(n, s) = (mn, ms + nr + rs).$$

While it is true that

$$\mathrm{GL}_\infty(R, I) = \mathrm{GL}_\infty(I^+, I),$$

independently of R, it is not generally true that $K_1(R, I) \stackrel{?}{=} K_1(I^+, I)$. In fact, the latter question is the subject of intense research!

10.2 Determinant of commutator

Let $D : \mathrm{GL}_\infty(R, I) \to G$ be a relative determinant homomorphism. We saw that after modding out by $\mathrm{E}_\infty(R, I)$, we get a homomorphism $\check{D} : K_1(R, I) \to G$ (with G abelian).

Let $U, V \in \mathrm{GL}_\infty(R)$.

- If U, V are both D-class, then

$$D(UVU^{-1}V^{-1}) = D(U)D(V)D(U)^{-1}D(V)^{-1} = 1, \qquad U, V \in \mathrm{GL}_\infty(R, I), \tag{10.10}$$

because of the homomorphism property of D and commutativity of G. This kind of triviality is familiar from absolute determinants.

The next examples reveal some interesting new features in the relative setting.

- If we only know that U is D-class, then U^{-1} is D-class, and so is the conjugate $VU^{-1}V^{-1}$. We still have

$$\begin{aligned} D(UVU^{-1}V^{-1}) &= D(U)D(VU^{-1}V^{-1}) \\ &= D(U)D(U^{-1}) \\ &= D(U)D(U)^{-1} = 1, \qquad U \in \mathrm{GL}_\infty(R, I). \end{aligned} \tag{10.11}$$

- If we only know that UV is D-class, then $VU = V(UV)V^{-1}$ is also D-class, as is $(VU)^{-1} = U^{-1}V^{-1}$. Then we still have

$$\begin{aligned} D(UVU^{-1}V^{-1}) &= D\big((UV)(VU)^{-1}\big) \\ &= D(UV)D(VU)^{-1} \\ &= D(UV)D(VUVV^{-1})^{-1} \quad \text{(conj. invaraince)} \\ &= D(UV)D(UV)^{-1} = 1, \quad UV \in \mathrm{GL}_\infty(R, I). \end{aligned} \tag{10.12}$$

- It is possible that $UVU^{-1}V^{-1}$ is D-class, without UV (or VU) being D-class. Now there is no reason for

$$D(UVU^{-1}V^{-1}) \stackrel{?}{=} 1, \qquad UVU^{-1}V^{-1} \in \mathrm{GL}_\infty(R, I). \tag{10.13}$$

In fact, it is known that $D(UVU^{-1}V^{-1})$ can take any value in $\mathbb{C}^\times$, see Example 10.14.

The last two cases should be compared to Remark 5 for traces.

Example 10.2. [Proofs ommitted.] A source of examples comes from the following result of Pincus–Helton–Howe,

$$\det(e^S e^T e^{-S} e^{-T}) = e^{\mathrm{Tr}[S,T]}, \qquad S, T \in \mathcal{L}(\mathcal{H}),\ [S, T] \in \mathcal{L}^1. \tag{10.14}$$

Here, Banach algebra techniques are needed to show that $[S,T] \in \mathcal{L}^1$ implies $e^S e^T e^{-S} e^{-T} \in \mathcal{D}(\mathcal{H})$. Then one establishes the identity for sS, tT for small s,t, and extends it to the general case by proving and invoking analyticity of the two expressions. Eq. (10.14) is the two-variable analogue of the more famous relationship

$$\det e^T = e^{\mathrm{Tr}\, T}, \qquad T \in \mathcal{L}^1, \tag{10.15}$$

that generalizes the obvious $n \times n$ matrix result.

Probably the simplest example of $0 \neq [S,T] \in \mathcal{L}^1$ is the pair of shift operators

$$S = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ 0 & 1 & 0 & \\ & 0 & \ddots & \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 & 0 & \\ & 0 & 1 & 0 \\ & & 0 & \ddots \end{pmatrix}, \qquad \text{on } \ell^2(\mathbb{N}),$$

which have $[S,T]$ being a rank-1 projection, thus $\mathrm{Tr}[S,T] = 1$. Then according to formula (10.14),

$$\det(e^{zS} e^T e^{-zS} e^T) = e^z, \qquad z \in \mathbb{C}.$$

Now, for a pair $U, V \in \mathrm{GL}_\infty(R)$ of invertibles,

$$\begin{aligned} UVU^{-1}V^{-1} \in \mathrm{GL}_\infty(R,I) &\Leftrightarrow UVU^{-1}V^{-1} - 1 = (UV - VU)U^{-1}V^{-1} \in M_\infty(I) \\ &\Leftrightarrow [U,V] \in M_\infty(I) \\ &\Leftrightarrow [q(U), q(V)] = 0 \in M_\infty(R/I) \end{aligned} \tag{10.16}$$

(We used the ideal property of I, and q is the quotient map $R \to R/I$, extended to matrix algebras.) In other words, $UVU^{-1}V^{-1}$ is D-class iff $q(U), q(V)$ are *commuting units* in $\mathrm{GL}_\infty(R/I)$.

Furthermore, the following calculation shows that $D(UVU^{-1}V^{-1})$ depends only on $q(U)$ and $q(V)$. Suppose $\tilde{U} \in \mathrm{GL}_\infty(R)$ satisfies $q(\tilde{U}) = q(U)$. Then criterion (10.16) shows that

$$\tilde{U}V\tilde{U}^{-1}V^{-1}, \qquad V(U\tilde{U}^{-1})V^{-1}(U\tilde{U}^{-1})^{-1}$$

are D-class. So we may calculate

$$\begin{aligned} D\big(\tilde{U}V\tilde{U}^{-1}V^{-1}\big) &= D\big((U\tilde{U}^{-1})(\tilde{U}V\tilde{U}^{-1}V^{-1})(\tilde{U}U^{-1})\big) && \text{(conj. invariance)} \\ &= D\big((UVU^{-1}V^{-1})(VU\tilde{U}^{-1}V^{-1}\tilde{U}U^{-1})\big) \\ &= D(UVU^{-1}V^{-1})\underbrace{D\big(V(U\tilde{U}^{-1})V^{-1}(U\tilde{U}^{-1})^{-1}\big)}_{=1 \text{ by } U\tilde{U}^{-1}\in \mathrm{GL}_\infty(R,I) \text{ and (10.11)}}. \end{aligned}$$

Therefore,

$$D(UVU^{-1}V^{-1}) = D(\tilde{U}V\tilde{U}^{-1}V^{-1}),$$

and similarly if V is replaced by $\tilde{V}$ satisfying $q(\tilde{V}) = q(V)$.

To summarize: $UVU^{-1}V^{-1}$ is D-class iff $q(U), q(V)$ are commuting units, in which case, $D(UVU^{-1}V^{-1})$ is already determined by the pair $q(U), q(V)$.

Thus, we anticipate that there is some "symbol" associated to commuting units $u, v \in \mathrm{GL}_\infty(R/I)$, denoted by

$$\{u, v\} \in K_2(R/I),$$

such that in particular, $\{q(U), q(V)\}$ determines $D(UVU^{-1}V^{-1})$. The invariant $\{q(U), q(V)\}$ would be even more primitive than $[UVU^{-1}V^{-1}]_{K_1(R,I)}$. The notation $K_2(R/I)$ is in anticipation of a "connecting homomorphism" $\partial : K_2(R/I) \to K_1(R, I)$, which maps the primitive invariant $\{q(U), q(V)\}$ to $[UVU^{-1}V^{-1}]_{K_1(R,I)}$.

The algebraic properties of $\{\,\cdot\,,\,\cdot\,\}$ should mirror those of $(U, V) \mapsto D(UVU^{-1}V^{-1})$. For example:

- Let $S \in \mathrm{GL}_\infty(R)$. Writing $\hat{U} = SUS^{-1}, \hat{V} = SVS^{-1}$, we get
$$D(\hat{U}\hat{V}\hat{U}^{-1}\hat{V}^{-1}) = D(UVU^{-1}V^{-1}).$$
- It is not hard to see that (exercise)
$$\begin{aligned} D(UVU^{-1}V^{-1}) &= D(VUV^{-1}U^{-1})^{-1} = D(U^{-1}VUV^{-1})^{-1} \\ D\big((U_1U_2)V(U_1U_2)^{-1}V^{-1}\big) &= D(U_1VU_1^{-1}V^{-1})D(U_2VU_2^{-1}V^{-1}). \end{aligned}$$

Correspondingly, $\{\cdot,\cdot\}$ should be bilinear and antisymmetric,

$$\{u, v\} = \{v, u\}^{-1}, \qquad \{u_1, v\}\{u_2, v\} = \{u_1u_2, v\}, \qquad u, v \in \mathrm{GL}_\infty(R/I) \text{ commute.}$$

We also expect that $\{u^{-1}, v\} = \{u, v\}^{-1}$. Furthermore, since $\{q(U), q(V)\}$ is meant to predict the abelian G-valued determinant D, the set $K_2(R/I)$ might be expected to be an *abelian group*.

We should keep the above relative determinant discussion in mind, as we construct the abelian group $K_2(-)$ and the symbol $\{\cdot, \cdot\} \in K_2(-)$ in the next section.