Lecture notes: Infinity Categories 2025

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1 Some categorical preliminaries

Abstracting the structural aspect of the study of linear maps of vector spaces, homomorphisms of groups/rings, continuous maps of topological spaces, etc., we get the concept of *category*.

1.1 Basic definitions

A *category* $\mathcal{C}$ comprises:

- A class $\mathrm{Ob}(\mathcal{C})$ of *objects*;
- For each pair of objects, a set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of *morphisms*, with associative composition law

$$\mathrm{Hom}_{\mathcal{C}}(Y,Z)\times \mathrm{Hom}_{\mathcal{C}}(X,Y) \stackrel{\circ}{\to} \mathrm{Hom}_{\mathcal{C}}(X,Z);$$

  Each object has a special *identity* morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$, such that

$$\mathrm{id}_X \circ f = f \circ \mathrm{id}_Y \qquad \forall\, f \in \mathrm{Hom}_{\mathcal{C}}(X,Y).$$

Technically, we have defined a *locally small* category. A *small* category has both $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ being sets.

Composition of morphisms is usually visualized in terms of arrows,

$$X \stackrel{f}{\to} Y \stackrel{g}{\to} Z.$$

The *opposite category* $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y,X) = \mathrm{Hom}_{\mathcal{C}}(X,Y),$$

and reversed composition law (arrows),

$$X \xleftarrow{f} Y \xleftarrow{g} Z.$$

	Objects	1-Morphisms	2-Morphisms
Set	Sets	Functions	
Top	Topological spaces	Continuous maps	
R-**Mod**	R-modules	Module homs	
Grp	Groups	Group homs	
Ab	Abelian groups $\equiv \mathbb{Z}$-modules	Abelian group homs	
AbMon	Abelian monoids	Abelian monoid homs	
Group G	Single object	Elements of G	
$\mathcal{D}^{\mathcal{C}}$ $\mathcal{D}_{\mathcal{C}}$	Functors $\mathcal{C} \to \mathcal{D}$ Functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$	Natural trans.	
Cat	Small categories	Functors	Natural trans.

Functors are maps of categories which respect composition and identity. Explicitly, a functor $F : \mathcal{C} \to \mathcal{D}$ comprises an object-level map,

$$F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D}),$$

as well as a morphism-level map,

$$F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)),$$

such that $F(g \circ f) = F(g) \circ F(f)$ and $F(\mathrm{Id}_{\mathcal{C}}) = \mathrm{Id}_{\mathcal{D}}$. Functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ are called *contravariant* functors $\mathcal{C} \to \mathcal{D}$.

Example 1.1. Consider the category $\mathbf{Top}_\bullet$ of pointed topological spaces (X, x_0) with basepoint-preserving continuous maps as morphisms. Fixing a space (X, x_0), one observes/proves that the set of homotopy classes of based loops in (X, x_0) forms a *group* $\pi_1(X, x_0)$ under the operation of concatenation, with the class of the constant loop being the identity group element. The group $\pi_1(X, x_0)$ is called the *fundamental group* of (X, x_0).

If a morphism

$$f : (X, x_0) \to (Y, y_0)$$

relates two pointed spaces (so $y_0 = f(x_0)$), then we also have a group homomorphism of the respective fundamental groups,

$$\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0), \tag{1.1}$$

i.e., a morphism in the category **Grp**. Namely, compose a representative based loop in (X, x_0) with f to get a based loop in (Y, y_0), then check that the resulting homotopy class is independent of the starting representative loop. Loop concatenation is respected, so we get a well-defined group homomorphism (1.1).

Furthermore, we also check/prove that whenever we can compose two morphisms, $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$, in $\mathbf{Top}_\bullet$, then

$$\pi_1(g \circ f) = \pi_1(g) \circ \pi_1(f)$$

holds in **Grp**. Of course,

$$\pi_1(\mathrm{id}_{(X,x_0)}) = \mathrm{id}_{\pi_1(X,x_0)}$$

as well. Therefore, π_1 is a *functor* from $\mathbf{Top}_\bullet$ to **Grp**.

There are many other *algebraic topology* functors taking topological categories to algebraic categories. Sometimes, we can go from algebra back to topology as well... but to understand whether going back-and-forth is "the same as doing nothing", we need to what it means to compare functors.

A morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a *isomorphism* (of the objects X, Y) if there is a morphism $\psi \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that

$$\psi \circ \phi = \mathrm{id}_X, \qquad \phi \circ \psi = \mathrm{id}_Y.$$

Since functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ can likewise be composed, we could say that F implements an "isomorphism of categories" if there is a G such that $G \circ F = \mathrm{Id}_{\mathcal{C}}$ and $F \circ G = \mathrm{Id}_{\mathcal{D}}$. While this is possible, it is more useful to consider the weaker notion of "equivalence of categories".

Example 1.2. Consider the category **fVect** of finite-dimensional vector spaces. There is a contravariant *dual space* functor $D : \mathbf{fVect} \to \mathbf{fVect}$ which assigns each vector space V to V^*. Also, each linear map $L : V \to W$ gets assigned to the dual linear map $D(L) = L^* : W^* \to V^*$.

For every object V, we know by dimension counting that $V^* = D(V)$ and $V = \mathrm{Id}_{\mathbf{fVect}}$ are isomorphic. However, there is no *canonical* choice of isomorphism in $\mathrm{Hom}_{\mathbf{fVect}}(V, V^*)$, unless, e.g., we choose bases.

There *is* a canonical isomorphism in $\mathrm{Hom}_{\mathbf{fVect}}(V, V^{**})$ — identify $v \in V$ with the evaluation map $\mathrm{ev}_V(v)$ (exercise). Selecting $\mathrm{ev}_V : V \xrightarrow{\cong} V^{**}$ for each V, is a *natural* choice at the category level, in the sense that it is compatible

with arbitrary morphisms $L : V \to W$ in **fVect**. That is, the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ \downarrow{\scriptstyle \mathrm{ev}_V} & & \downarrow{\scriptstyle \mathrm{ev}_W} \\ V^{**} & \xrightarrow{L^{**}} & W^{**} \end{array},$$

for all morphisms $V \xrightarrow{L} W$ in **fVect**. (Exercise.)

The top horizontal arrow is the identity functor $\mathrm{Id}_{\mathbf{fVect}} : \mathbf{fVect} \to \mathbf{fVect}$ applied to the morphism $V \xrightarrow{L} W$, and bottom horizontal arrow is the double-dual functor $D^2 : \mathbf{fVect} \to \mathbf{fVect}$ applied to the same morphism $V \xrightarrow{L} W$. The resulting arrows are connected by the "natural isomorphisms" ev_V and $\mathbf{ev}_W$.

In general, we may compare two functors $F, G : \mathcal{C} \to \mathcal{D}$ as follows. A *natural transformation* $\eta : F \Rightarrow G$ is a collection of morphisms $\{\eta_X : F(X) \to G(X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ in $\mathcal{D}$, such that

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\varphi)} & F(Y) \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow{G(\varphi)} & G(Y) \end{array}.$$

commutes for every morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. (If F, G are contravariant functors, the horizontal arrows should be reversed in the above diagram.)

If $\psi : Y \to Z$ is another morphism, we have the commuting diagram

$$\begin{array}{ccccc} & & F(\psi\circ\phi) & & \\ & \overset{\frown}{} & & \overset{\frown}{} & \\ F(X) & \xrightarrow{F(\varphi)} & F(Y) & \xrightarrow{F(\psi)} & F(Z) \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} & & \downarrow{\scriptstyle \eta_Z} \\ G(X) & \xrightarrow{G(\varphi)} & G(Y) & \xrightarrow{G(\psi)} & G(Z) \\ & \underset{\smile}{} & & \underset{\smile}{} & \\ & & G(\psi\circ\phi) & & \end{array}.$$

As F, G are functors, the top and bottom caps commute. A natural transformation $\eta : F \Rightarrow G$ *respects this arrow-respecting property* of F, G. This prompts us to think of a natural transformation as a "morphism of functors".

This terminology is justified, because we can (associatively) compose a natural transformation $\eta : F_1 \Rightarrow F_2$ with another natural transformation $\varepsilon : F_2 \Rightarrow F_3$,

$$\begin{array}{ccc} F_1(X) & \xrightarrow{F_1(\varphi)} & F_1(Y) \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\ F_2(X) & \xrightarrow{F_2(\varphi)} & F_2(Y) \\ \downarrow{\scriptstyle \varepsilon_X} & & \downarrow{\scriptstyle \varepsilon_Y} \\ F_3(X) & \xrightarrow{F_3(\varphi)} & F_3(Y) \end{array},$$

to obtain a natural transformation $F_1 \Rightarrow F_3$. Of course, every functor F has the identity natural transformation $\mathrm{Id}_F : F \Rightarrow F$.

Therefore, once we fix categories $\mathcal{C}, \mathcal{D}$, we can consider the class of functors $F : \mathcal{C} \to \mathcal{D}$ itself as a category, denoted $\mathcal{D}^{\mathcal{C}}$. For the category of contravariant functors, we write $\mathcal{D}_{\mathcal{C}}$ instead. A morphism between two such fuctors F, G is, by definition, a natural transformation $\eta : F \Rightarrow G$. We often write

$$\mathrm{Nat}(F, G) = \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G), \qquad F, G \in \mathrm{Ob}(\mathcal{D}^{\mathcal{C}}). \tag{1.2}$$

A technicality: generally, some assumptions are needed to ensure that the $\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$ are actual sets.

Some authors call an ordinary category a "1-category". There is a 1-category **Cat** whose objects are small categories, and whose hom-sets are the various $\mathcal{D}^{\mathcal{C}}$ (which are sets). Since we have seen that each $\mathcal{D}^{\mathcal{C}}$ is itself a 1-category, we may "enrich" **Cat** by acknowledging that its hom-sets possess this extra structure. Then **Cat** is sometimes called a "2-category", with the natural transformations being the "2-morphisms".

There are some extra details needed to "strictly" define what "2-categories" and their even higher versions are. We will not need to go into these details.

A morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a *isomorphism* (of the objects X, Y) if there is a morphism $\psi \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that

$$\psi \circ \phi = \mathrm{id}_X, \qquad \phi \circ \psi = \mathrm{id}_Y.$$

Since functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ can likewise be composed, we might consider F to implement an "isomorphism of categories" if there is a G such

that $G \circ F = \mathrm{Id}_{\mathcal{C}}$ and $F \circ G = \mathrm{Id}_{\mathcal{D}}$. While this is possible, it is more useful to consider the weaker notion of "equivalence of categories", as defined below.

If a natural transformation $\eta : F \Rightarrow G$ is such that all the η_X are isomorphisms, we say that F, G are *naturally isomorphic*. For example, we have seen that the double-dual functor D^2 is naturally isomorphic to $\mathrm{Id}_{\mathbf{fVect}}$, via the evaluation isomorphisms.

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to implement an *equivalence* of categories, if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that FG is naturally isomorphic to $\mathrm{Id}_{\mathcal{D}}$ and GF is naturally isomorphic to $\mathrm{Id}_{\mathcal{C}}$. Note: we do not demand $G(F(X)) = X$ for all objects X, but only $G(F(X)) \cong X$ naturally in X.

Here is a surprising equivalence of categories: The category $\mathbf{UComC}^*$ of unital commutative C^*-algebras is equivalent to the (opposite of the) category $\mathbf{CH}^{\mathrm{op}}$ of compact Hausdorff spaces.

"Surprising" means that the categories involved appear, at first sight, to have very different natures. As $\mathbf{UComC}^*$ is an *analytic* (not pure algebraic!) concept, the functor implementing its equivalence with $\mathbf{CH}$ must involve some (deep) theorems in analysis.

1.2 Adjoints

We had discussed how *isomorphism* of categories suffers from focussing too much on object-level correspondences. Relaxing *isomorphism* to *equivalence* of categories (implemented by $F : \mathcal{C} \to \mathcal{D}$), we still have bijections on hom-sets (depending naturally on the objects), but objects themselves are treated up-to-isomorphism (see Theorem 1.1). We can further relax the notion of equivalence further, while still bijectively and naturally respecting the hom-sets. The key is to consider a *pair* of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$.

An *adjunction* between functors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ is a natural isomorphism between

$$\mathrm{Hom}_{\mathcal{D}}(L-, -), \qquad \mathrm{Hom}_{\mathcal{C}}(-, R-),$$

as functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$. In this case, we write $L \dashv R$.

By definition, an adjunction is a collection of bijections

$$\begin{aligned} \overline{(-)} : \mathrm{Hom}_{\mathcal{D}}(Lc, d) &\cong \mathrm{Hom}_{\mathcal{C}}(c, Rd) \\ f &\mapsto \bar{f}, \qquad\qquad \forall\, c \in \mathrm{Ob}(\mathcal{C}),\, d \in \mathrm{Ob}(\mathcal{D}), \end{aligned}$$

depending naturally on (c, d). (This equation superficially resembles the equation defining the adjoint of operators on Hilbert spaces.)

Explicitly, naturality of the $\overline{(-)}$ means commutativity of

$$\begin{array}{ccc} \big(Lc \xrightarrow{f} d\big) & \xrightarrow{b_* \circ L(a)^*} & \big(Lc' \xrightarrow{b \circ f \circ L(a)} d'\big) \\ \Big\downarrow{\scriptstyle \overline{(-)}} & & \vdots\, {\scriptstyle \overline{(-)}} \\ \big(c \xrightarrow{\bar{f}} R(d)\big) & \xrightarrow{R(b)_* \circ a^*} & \big(c' \to R(d')\big) \end{array} \quad , \qquad \forall\, d \xrightarrow{b} d',\ c' \xrightarrow{a} c,$$

i.e.,

$$\overline{b \circ f \circ L(a)} = R(b) \circ \bar{f} \circ a, \qquad \forall\, d \xrightarrow{b} d',\ c' \xrightarrow{a} c. \tag{1.3}$$

Example 1.3. Let $R : \mathbf{Vect} \to \mathbf{Set}$ be the "forgetful" functor taking a vector space to its underlying set. Let $L : \mathbf{Set} \to \mathbf{Vect}$ be the "free" functor taking a set S to the vector space with basis $\{e_s \ : \ s \in S\}$. (L also assigns a set function $S' \xrightarrow{a} S$ to the linear map taking $e_{s'}$ to $e_{a(s')}$.) For each set S and vector space V, we take

$$\begin{aligned} \overline{(-)} : \mathrm{Hom}_{\mathbf{Vect}}(LS, V) &\to \mathrm{Hom}_{\mathbf{Set}}(S, RV) \\ f &\mapsto \big(s \mapsto f(e_s)\big). \end{aligned}$$

To see that $\overline{(-)}$ is a bijection, check that

$$\begin{aligned} \overline{(-)}^{-1} : \mathrm{Hom}_{\mathbf{Set}}(S, RV) &\to \mathrm{Hom}_{\mathbf{Vect}}(LS, V) \\ g &\mapsto \Big(\sum_{s \in S} \lambda_s e_s \mapsto \sum_{s \in S} \lambda_s g(s)\Big) \end{aligned}$$

is its inverse. Furthermore, it is not hard to check that the naturality (1.3) is satisfied by $\overline{(-)}$.

Example 1.4. Relevant to K_1-theory: The abelianization functor $L : \mathbf{Grp} \to \mathbf{Ab}$ is left-adjoint to the inclusion functor $R : \mathbf{Ab} \to \mathbf{Grp}$. (Exercise)

Example 1.5. Relevant to K_0-theory: The Grothendick group construction is left-adjoint to the forgetful functor from abelian groups to abelian monoids. See Section 1.2.3.

Example 1.6. Within homotopy theory, there are so-called *suspension* $S(-)$ and *loop space* $\Omega(-)$ functors on the homotopy category (roughly, topological spaces modded out by homotopy). Then S is left-adjoint to Ω.

Example 1.7. Here is a more compelling example which bridges apparently different kinds of categories: We had mentioned that compact Hausdorff spaces are (anti-)equivalent to unital commutative C^*-algebras. In particular, there is an adjunction between these categories (see Theorem 1.1). In fact, there is an adjunction between topological spaces and unital commutative Banach algebras (which is not an equivalence).

1.2.1 Unit-counit

Given $L \dashv R$, we may use the particular bijections

$$\begin{aligned} \overline{(-)} &: \mathrm{Hom}_{\mathcal{D}}(Lc, Lc) \cong \mathrm{Hom}_{\mathcal{C}}(c, RLc), && c \in \mathrm{Ob}(\mathcal{C}), \\ \overline{(-)} &: \mathrm{Hom}_{\mathcal{D}}(LRd, d) \cong \mathrm{Hom}_{\mathcal{C}}(Rd, Rd), && d \in \mathrm{Ob}(\mathcal{D}), \end{aligned}$$

to define morphisms

$$\begin{aligned} \beta_c &:= \overline{\mathrm{id}_{Lc}}, \quad c \in \mathrm{Ob}(\mathcal{C}), \\ \overline{\alpha_d} &:= \mathrm{id}_{Rd}, \quad d \in \mathrm{Ob}(\mathcal{D}). \end{aligned} \tag{1.4}$$

It follows from the naturality property (1.3) that these $\beta_c : c \to RLc$ and $\alpha_d : LRd \to d$ respectively assemble into natural transformations,

$$\alpha : LR \Rightarrow \mathrm{Id}_{\mathcal{D}}, \qquad \beta : \mathrm{Id}_{\mathcal{C}} \Rightarrow RL, \tag{1.5}$$

respectively called the *counit* and *unit* of the adjunction. (Exercise)

Example 1.8. For the adjunction in Example 1.3, the unit is given by the collection of functions

$$\begin{aligned} \beta_S : S &\mapsto RL(S) \\ s &\mapsto R(e_s), \end{aligned}$$

i.e., the embedding of S as a basis for the vector space it generates (the latter vector space regarded as a set.) Work out the counit in this case.

We only used part of the data of the adjunction when constructing its counit/unit α, β. There is a further relationship between α and β, which encodes the remaining data. To state this relationship, we need some preparatory notation. In general, when $\alpha : LR \Rightarrow \mathrm{Id}_{\mathcal{D}}$ and $\beta : \mathrm{Id}_{\mathcal{C}} \Rightarrow RL$ are natural transformations (not necessarily coming from an adjunction), we write $L\beta$ for the natural transformation

$$L\beta : \underbrace{L}_{L \circ \mathrm{Id}_{\mathcal{C}}} \Rightarrow \underbrace{LRL}_{L \circ (RL)}, \qquad (L\beta)_c = L(\beta_c),$$

and similarly,

$$\begin{aligned} \alpha L : LRL \Rightarrow L, &\qquad (\alpha L)_{LRLc} = \alpha_{Lc}, \\ \beta R : R \Rightarrow RLR, &\qquad (\beta R)_d = \beta_{Rd}, \\ R\alpha : RLR \Rightarrow R\alpha, &\qquad (R\alpha)_{RLRd} = R(\alpha_d). \end{aligned}$$

The counit/unit α, β of an adjunction $L \dashv R$ satisfy the *triangle identities*

$$\begin{array}{ccc} L & \overset{L\beta}{\Longrightarrow} & LRL \\ & \underset{\mathrm{Id}_L}{\searrow} & \Downarrow \alpha L \\ & & L \end{array}, \qquad \begin{array}{ccc} RLR & \overset{\beta R}{\Longleftarrow} & R \\ {\scriptstyle R\alpha}\Downarrow & \swarrow_{\mathrm{Id}_R} & \\ R & & \end{array}, \qquad \text{commute,} \tag{1.6}$$

by virtue of the naturality (1.3). (Exercise)

Remark 1. Conversely, if $\alpha : LR \Rightarrow \mathrm{Id}_{\mathcal{D}}$ and $\beta : \mathrm{Id}_{\mathcal{C}} \Rightarrow RL$ satisfy the triangle identities (1.6), then we can construct an adjunction $L \dashv R$ whose counit/unit is α, β. (Exercise)

Remark. Suppose $L : \mathcal{C} \to \mathcal{D}$ implements an equivalence of categories. By definition, there exists a functor $R : \mathcal{D} \to \mathcal{C}$ as well as natural *isomorphisms*

$$\alpha : LR \cong \mathrm{Id}_{\mathcal{D}}, \qquad \beta : \mathrm{Id}_{\mathcal{C}} \cong RL.$$

This, however, does not mean that α, β satisfy the triangle identities (1.6), so it is unclear that there is an adjunction $L \dashv R$. Nevertheless, we can always "improve" the data R, α, β to achieve an adjunction, based on the following result.

Theorem 1.1. *Let $L : \mathcal{C} \to \mathcal{D}$ be a functor. The following are equivalent (Exercise):*

1. *L implements an equivalence of categories.*
2. *L is bijective on hom-sets, and essentially surjective.*[1].
3. *There exists a functor $R : \mathcal{D} \to \mathcal{C}$ such that $L \dashv R$ and has unit/counit being natural isomorphisms.*

In fact, the proof (omitted) will reveal that there is a suitable modification α' of the natural isomorphism α, such that the triangle identities are achieved. By Remark 1, this means that $L \dashv R$ with counit/unit being α', β.

[1]Every object of $\mathcal{D}$ is isomorphic to something in the range of L.

1.2.2 Universal property

Theorem 1.2. *For a functor $R : \mathcal{D} \to \mathcal{C}$, the following are equivalent.*

1. *R has a left-adjoint.*
2. *Each $c \in \mathrm{Ob}(\mathcal{C})$ has an associated $Lc \in \mathrm{Ob}(\mathcal{D})$ and a universal morphism $c \xrightarrow{\beta_c} RLc$.*

Here, "universal" means that given any morphism $c \xrightarrow{\varphi} Rd$, there exists a unique $Lc \xrightarrow{\tilde{\varphi}} d$ making the following commute:

$$\begin{array}{ccc} c & \xrightarrow{\beta_c} & RLc \\ & \searrow^{\varphi} & \downarrow{\scriptstyle R(\tilde{\varphi})} \\ & & Rd \end{array} . \tag{1.7}$$

Proof. Let L be a left adjoint of R. The unit $\beta : \mathrm{Id}_{\mathcal{C}} \Rightarrow RL$ of the adjunction provides morphisms $\beta_c : c \to RLc$ (Eq. (1.4)). Let us check that these β_c are universal. So suppose $\varphi \in \mathrm{Hom}_{\mathcal{C}}(c, Rd)$ is given. We need to find $Lc \xrightarrow{\tilde{\varphi}} d$ such that

$$R(\tilde{\varphi}) \circ \beta_c = \varphi. \tag{1.8}$$

By the general naturality property (1.3), the left side is

$$R(\tilde{\varphi}) \circ \beta_c = R(\tilde{\varphi}) \circ \overline{\mathrm{id}_{Lc}} = \overline{\tilde{\varphi} \circ \mathrm{id}_{Lc}} = \overline{\tilde{\varphi}}.$$

As $\overline{(-)}$ is a bijection, there is a *unique* solution to (1.8). This verifies that β_c is universal.

Conversely, assume that each $c \in \mathrm{Ob}(\mathcal{C})$ has an associated $Lc \in \mathrm{Ob}(\mathcal{D})$ and a universal morphism $c \xrightarrow{\beta_c} RLc$. Below, we will argue that L can be promoted to a functor $\mathcal{C} \to \mathcal{D}$ such that $L \dashv R$.

- Assign each morphism $c \xrightarrow{a} c'$ to the unique morphism

$$L(a) := \widetilde{\beta_{c'} \circ a}, \tag{1.9}$$

  provided by the universality of β_c. In other words, $L(a)$ is the unique morphism making

$$\begin{array}{ccc} c & \xrightarrow{\beta_c} & RLc \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle R(L(a))} \\ c' & \xrightarrow{\beta_{c'}} & RLc' \end{array} \tag{1.10}$$

commmute.

- We have
$$L(\mathrm{id}_c) = \widetilde{\beta_c \circ \mathrm{id}_c} = \widetilde{\beta}_c = \mathrm{id}_{Lc}.$$
For a composition $c \xrightarrow{a_1} c' \xrightarrow{a_2} c''$, the following diagram commutes
$$\begin{array}{ccc} c & \xrightarrow{\beta_c} & RLc \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle R(L(a_1))} \\ c' & \xrightarrow{\beta_{c'}} & RLc' \\ {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle R(L(a_2))} \\ c'' & \xrightarrow{\beta_{c''}} & RLc'' \end{array} .$$
Consider just the outer commuting rectangle,
$$\begin{array}{ccc} c & \xrightarrow{\beta_c} & RLc \\ {\scriptstyle a_2\circ a_1}\downarrow & & \downarrow{\scriptstyle R(L(a_2)\circ L(a_1))} \\ c''R & \xrightarrow{\beta_{c''}} & Lc'' \end{array} .$$
At the same time, $L(a_2 \circ a_1)$ is, by definition, the unique map such that
$$\begin{array}{ccc} c & \xrightarrow{\beta_c} & RLc \\ {\scriptstyle a_2\circ a_1}\downarrow & & \downarrow{\scriptstyle R(L(a_2\circ a_1))} \\ c''R & \xrightarrow{\beta_{c''}} & Lc'' \end{array}$$
Therefore, $L(a_2 \circ a_1) = L(a_2) \circ L(a_1)$, which means that L is indeed a functor.

- Finally, based on the diagram (1.7), we guess that the bijections
$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(Lc, d) &\cong \mathrm{Hom}_{\mathcal{C}}(c, Rd) \\ f &\mapsto \bar{f} := R(f) \circ \beta_c \\ \tilde{\varphi} &\mapsfrom \varphi \end{aligned} \tag{1.11}$$
specify an adjunction $L \dashv R$. so it defines bijections. What needs to be checked is that the above $\overline{(-)}$ satisfies the naturality property (1.3) (Exercise).

□

1.2.3 Grothendieck construction

The following construction illustrates Theorem 1.2, and it is fundamental in K_0-theory.

Let $M \equiv (M, +)$ be an abelian monoid. On $M \times M$, define the equivalence relation

$$(x_1, y_1) \sim (x_2, y_2) \text{ iff } \exists z \in M \ : \ x_1 + y_2 + z = x_2 + y_1 + z. \tag{1.12}$$

The *Grothendieck group* $G(M)$ is the set $(M \times M)/\sim$, with composition rule

$$[x_1, y_1] + [x_2, y_2] := [x_1 + x_2, y_1 + y_2], \tag{1.13}$$

together with the monoid homomorphism

$$\iota_M : M \to G(M), \qquad x \mapsto [x, 0]. \tag{1.14}$$

Check that (Exercise):

- (1.13) is well-defined on equivalence classes.
- The zero element of $G(M)$ is the equivalence class $[z, z]$, where $z \in M$ is arbitrary.
- The inverse of $[x, y]$ is $[y, x]$.
- ι_M is indeed a monoid homomorphism.

Theorem 1.3. *Let M be an abelian monoid.*

- *For any abelian group H and monoid homomorphism $\varphi : M \to H$, there exists a unique group homomorphism $\tilde{\varphi} : G(M) \to H$ such that the following commutes*

$$\begin{array}{ccc} M & \xrightarrow{\iota_M} & G(M) \\ & \searrow^{\varphi} & \downarrow \tilde{\varphi} \\ & & H \end{array} \quad . \tag{1.15}$$

- *If $\varphi : M \to M'$ is an abelian monoid homomorphism, there is a unique abelian group homomorphism $G(\varphi) : G(M) \to G(M')$ such that the following commutes,*

$$\begin{array}{ccc} M & \xrightarrow{\iota_M} & G(M) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle G(\varphi)} \\ M' & \xrightarrow{\iota_{M'}} & G(M') \end{array} . \tag{1.16}$$

Proof. Given φ, we check that

$$\tilde{\varphi}[x, y] = \varphi(x) - \varphi(y) \tag{1.17}$$

is the unique abelian group homomorphism which makes (1.15) commute.

Because of the universal property (1.15) of $M \xrightarrow{\iota_M} G(M)$, there is a unique $G(\varphi) : G(M) \to G(M')$ making (1.16) commute. □

In fact, replacing φ by $\iota_{M'} \circ \varphi$ in formula (1.17), the $G(\varphi)$ are given by the formula

$$\begin{aligned} G(\varphi)[x, y] = \widetilde{\iota_{M'} \circ \varphi}[x, y] &= \iota_{M'} \circ \varphi(x) - \iota_{M'} \circ \varphi(y) \\ &= [\varphi(x), 0] - [\varphi(y), 0] \\ &= [\varphi(x), \varphi(y)]. \end{aligned}$$

If follows easily that G is actually a *functor*

$$G : \mathbf{AbMon} \to \mathbf{Ab},$$

where **AbMon** denotes the category of abelian monoids.

As written, the universal property (1.15) of $M \xrightarrow{\iota_M} G(M)$ is a little strange, since it mixes up the categories **AbMon** and **Ab**. Let $F : \mathbf{Ab} \to \mathbf{AbMon}$ denote the forgetful functor (forget the inverse structure). Then restate (1.15) as follows:

Proposition 1.4. *Let $M \in \mathrm{Ob}(\mathbf{AbMon})$. For any $H \in \mathrm{Ob}(\mathbf{Ab})$ and any $\varphi \in \mathrm{Hom}_{\mathbf{AbMon}}(M, F(H))$, there exists a unique $\tilde{\varphi} \in \mathrm{Hom}_{\mathbf{Ab}}(G(M), H)$ making the following diagram commute,*

$$\begin{array}{ccc} M & \xrightarrow{\iota_M} & FG(M) \\ & \searrow^{\varphi} & \vdots \\ & & \exists!\, F(\tilde{\varphi}) \\ & & F(H) \end{array} .$$

By Theorem 1.2 (particularly, formula (1.11)), the correspondence

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Ab}}(G(M), H) &\cong \mathrm{Hom}_{\mathbf{AbMon}}(M, F(H)) \\ \tilde{\varphi} &\leftrightarrow \varphi \end{aligned}$$

gives an adjunction $G \dashv F$, and $\iota \equiv \{\iota_M\}_{M \in \mathrm{Ob}(\mathbf{AbMon})}$ is precisely the unit for this adjunction.

Remark. An abelian monoid S has the *cancellation* property if

$$x + z = y + z \;\Rightarrow\; x = y, \qquad x, y, z \in S.$$

This property fails for general abelian monoids, so the appearance of z in the definition (1.12) is not redundant. The failure of cancellation is related to non-injectivity of ι_M. Contrast with Example 1.8, where the unit comprises injective maps for all $S \in \mathrm{Ob}(\mathbf{Sets})$.