7 Algebraic K_1

7.1 h-Cobordism

Let M_0, M_1 be closed smooth n-manifolds,

- A *cobordism* from M_0 to M_1 is a compact $(n+1)$-manifold-with-boundary W such that $\partial W = \partial_0 W \sqcup \partial_1 W$ is a disjoint union, together with diffeomorphisms $f_i : M_i \xrightarrow{\cong} \partial_i W$, $i = 0, 1$.

  If (W, f_0, f_1) is a cobordism from M_0 (to M_1) and (W', f_0', f_1') is a cobordism from M_0 (to M_1'), we say that they are *diffeomorphic rel.* M_0 if there is a diffeomorphism $F : W \to W'$ such that $F \circ f_0 = f_0'$.

- A cobordism (W, f_0, f_1) from M_0 to M_1 is called a *h-cobordism* if $\partial_i W \hookrightarrow W$, $i = 0, 1$, are homotopy equivalences.

The most "trivial" cobordism from M_0 to another manifold is the cylinder, $W = M_0 \times [0, 1]$. Here, the obvious diffeomorphisms

$$M_0 \to M_0 \times \{i\}, \qquad M_1 = M_0 \to M_0 \times \{1\},$$

are implicit.

Theorem 7.1 (s-Cobordism Theorem, Barden–Mazur–Stallings 1963)**.** *Let M_0 be a connected closed smooth n-manifold, $n \geq 5$. A h-cobordism from M_0 is diffeomorphic rel. M_0 to $M_0 \times [0, 1]$, iff its Whitehead torsion vanishes.*

Theorem 7.1 is difficult, and we will not prove it[11]. We merely mention that s stands for "simple homotopy equivalence", and that the Whitehead torsion of a h-cobordism from M_0 measures how the homotopy equivalence $\partial_0 W \to W$ fails to be simple. This failure is related to the non-triviality of the h-cobordism in a rather intricate way. Quite remarkably, these Whitehead torsion obstructions are elements of the so-called *Whitehead group* $\mathrm{Wh}(\pi_1(M_0))$, depending only on the fundamental group of M_0 (a topological, not smooth, invariant).

In Section 7.4, we will define the Whitehead group $\mathrm{Wh}(G)$ of a group G, and also see that $\mathrm{Wh}(\{1\}) = \{1\}$. So if M_0 is simply-connected, Whitehead torsion obstructions automatically vanish. Thus the s-cobordism theorm implies:

[11] See Chapters 2-3 of the recent book *Surgery theory: foundations* by W. Lück and T. Macko for a proof.

Theorem 7.2 (h-Cobordism Theorem, Smale, 1962)**.** *Let M_0 be a simply-connected closed smooth n-manifold, $n \geq 5$. Every h-cobordism from M_0 is diffeomorphic rel. M_0 to $M_0 \times [0,1]$.*

One indication of the significance of the h-cobordism theorem is that it leads quite readily to a proof the the generalized Poincaré conjecture in dimensions ≥ 6 (due to Smale, 1961, who proved it in dimensions ≥ 5). Here is a sketch of the argument:

- Let Σ^n be a closed smooth manifold which is homotopy equivalent to S^n. Let W be Σ^n with two disjoint open balls B_0^n and B_1^n removed. Then $\partial W = \partial_0 W \sqcup \partial_1 W$ with $\partial_i W = \partial B_i^n$, and W is a h-cobordism from $\partial_0 W = \partial B_0^n$.

- Since $\partial_0 W = \partial B_0^n$ is an $(n-1)$-sphere of dimension ≥ 5, it is simply-connected, and the h-cobordism theorem applies to give a diffeomorphism $F : \partial B_0^n \times [0,1] \to W$ rel. ∂B_0^n. In particular, we get a diffeomorphism $f_1 : \partial B_0^n \times \{1\} \to \partial_1 W = \partial B_1^n$, which we extend to a homeomorphism $\tilde{f}_1 : B_0^n \times \{1\} \to B_1^n$ by the "Alexander trick",

$$\tilde{f}_1(r\theta) = r f_1(\theta), \qquad r \in [0,1],\, \theta \in \partial B_0^n \cong S^{n-1}.$$

- Altogether, we get a homeomorphism from

$$(B_0^n \times \{0\}) \cup_{\partial B_0^n \times \{0\}} (\partial B_0^n \times [0,1]) \cup_{\partial B_0^n \times \{1\}} (B_0^n \times \{1\})$$

  to

$$B_0^n \cup_{\partial B_0^n} W \cup_{\partial B_1^n} B_1^n = \Sigma^n,$$

  via $\mathrm{id} \cup F \cup \tilde{f}_1$. The first space is homeomorphic to an n-sphere. Thus Σ^n is homeomorphic to the n-sphere.

7.2 The $K_1(-)$ functor

Throughout this subsection, R denotes a unital ring, $\mathrm{M}_n(R)$ denotes the ring of $n \times n$ matrices with entries in R, and $\mathrm{GL}_n(R)$ denotes the group of invertible elements in $\mathrm{M}_n(R)$.

We also write $\mathrm{M}_\infty(R)$ for the ring of $\mathbb{N} \times \mathbb{N}$ matrices with entries in R, with only finitely many non-zero entries. In Example 4.2, we saw that $\mathrm{M}_\infty(R)$, with

the upper-left inclusions $\mathrm{M}_n(R) \hookrightarrow \mathrm{M}_\infty(R)$, $a \mapsto \mathrm{diag}(a, 0_\infty)$, is the colimit of the diagram of upper-left inclusions $\mathrm{M}_m(R) \hookrightarrow \mathrm{M}_n(R), m \leq n$. Here, note that $\mathrm{M}_\infty(R)$ is a non-unital ring, so we are taking the colimit in the category of possibly non-unital rings. Because the indexing category arises from the directed set $(\mathbb{N}, \leq)$, and one sometimes says "direct limit" instead of "colimit".

Similarly, the embeddings

$$\begin{aligned} \mathrm{GL}_m(R) &\hookrightarrow \mathrm{GL}_n(R) \\ U &\mapsto \begin{pmatrix} U & 0 \\ 0 & 1_{n-m} \end{pmatrix}, \qquad\qquad m \leq n, \end{aligned}$$

can be organized into a diagram $\mathbb{N} \to \mathbf{Grp}$, with colimit being

$$\mathrm{GL}_\infty(R) := \bigcup_{n \geq 1} \mathrm{GL}_n(R),$$

equipped with the obvious inclusions $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_\infty(R)$ taking U to $\mathrm{diag}(U, 1_\infty)$.

Definition 4. $K_1(-)$ is the functor from the category **URings** of unital rings to the category **Ab** of abelian groups, taking a unital ring R to the abelianization of $\mathrm{GL}_\infty(R)$, and taking a unital ring homomorphism $\phi : R \to R'$ to the abelian group homomorphism

$$\begin{aligned} \phi_* : K_1(R) &\to K_1(R') \\ [U] &\mapsto [\phi(U)]. \end{aligned}$$

For a group G, its *commutator subgroup* $[G, G]$ is the subgroup generated by the multiplicative commutators $ghg^{-1}h^{-1}, g, h \in G$. One checks that $[G, G]$ is normal, and that the quotient group $G_{\mathrm{ab}} := G/[G, G]$ is abelian. Furthermore, any group homomorphism $\varphi : G \to H$ with H abelian factors uniquely through G_{ab},

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ {\scriptstyle \mathrm{mod}\,[G,G]}\downarrow & \nearrow_{\exists!\,\check{\varphi}} & \\ G_{\mathrm{ab}} & & \end{array} \qquad , \qquad \check{\varphi}[g] = \varphi(g). \tag{7.1}$$

Applying this to $\varphi : \mathrm{GL}_\infty(R) \xrightarrow{\phi} \mathrm{GL}_\infty(R') \xrightarrow{\mathrm{ab}} K_1(R')$, we get the map $\phi_* := \check{\varphi}$ in Definition 4.

The abelianization of $\mathrm{GL}_\infty(R)$ turns out to have a convenient description, as explained below.

First, for $r \in R$ and distinct $i, j \in \mathbb{N}$, the *elementary matrix* $e_{ij}(r)$ is the element of $\mathrm{GL}_\infty(R)$ which has 1s along the diagonal, r in the (ij)-th entry, and zeroes elsewhere. Such elementary matrices satisfy

$$e_{ij}(r)e_{ij}(s) = e_{ij}(r+s), \tag{7.2}$$

and, in particular,

$$e_{ij}(r)^{-1} = e_{ij}(-r). \tag{7.3}$$

There are other "commutation" relations, such as

$$e_{ij}(r)e_{lk}(s) = e_{lk}(r)e_{ij}(s), \qquad i \neq k, j \neq l, \tag{7.4}$$

$$e_{ij}(r)e_{jk}(s) = e_{ik}(rs) \cdot e_{jk}(s)e_{ij}(r), \qquad i \neq k. \tag{7.5}$$

So elementary matrices commute unless their indices match up in accordance with matrix multiplication rules.

Remark 4. For $T \in \mathrm{M}_\infty(R)$, the matrices

$$e_{ij}(r)T, \qquad Te_{ij}(r)$$

are obtained from T by adding $r \cdot (j\text{-th row})$ to the i-th row, or $r \cdot (i\text{-th column})$ to the j-th column, respectively.

We write $\mathrm{E}_n(R)$ for the subgroup generated by elementary matrices in $\mathrm{GL}_n(R) \subseteq \mathrm{GL}_\infty(R)$. Then we have a diagram $\mathrm{E}_m(R) \hookrightarrow \mathrm{E}_n(R), m \leq n$, and we can take

$$\mathrm{E}_\infty(R) := \bigcup_{n \geq 1} \mathrm{E}_n(R)$$

to be the colimit.

Lemma 7.3 (Whitehead's Lemma)**.** *For any $u \in \mathrm{GL}_n(R)$, the matrix $\begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}$ lies in $\mathrm{E}_{2n}(R)$. Consequently, $\mathrm{E}_\infty(R)$ is the commutator subgroup of $\mathrm{GL}_\infty(R)$, and*

$$K_1(R) = \mathrm{GL}_\infty(R)/\mathrm{E}_\infty(R).$$

Proof. First, we have the factorization

$$\begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ U^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-U \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-U^{-1} \\ 0 & 1 \end{pmatrix}$$

into a product of upper/lower block-triangular matrices. So it will suffice to show that any $m \times m$ upper/lower triangular matrix with 1s along the diagonal lies in $\mathrm{E}_m(R)$.

So, let $T = (T_{ij}) \in \mathrm{GL}_m(R)$ be upper triangular, with $T_{ii} = 1$, $\forall\, i$. The following matrix

$$T^{(1)} := T e_{12}(-T_{12}) e_{23}(-T_{23}) \cdots e_{m-1,m}(-T_{m-1,m})$$

remains upper triangular with 1s along the diagonal. Furthermore, its first upper super-diagonal vanishes ($T^{(1)}_{i,i+1} = 0$). Next, define

$$T^{(2)} := T^{(1)} e_{13}(-T^{(1)}_{13}) e_{24}(-T^{(1)}_{24}) \cdots e_{m-2,m}(-T^{(1)}_{m-2,m}),$$

which remains upper triangular with 1s along the diagonal, and has vanishing first and second upper super-diagonals, $T^{(2)}_{i,i+1} = 0$ and $T^{(2)}_{i,i+2} = 0$. Continue this process until we arrive at the upper triangular matrix $T^{(n-1)}$ with 1s along the diagonal, and vanishing 1-st to $(n-1)$-th upper super-diagonals; i.e., $T^{(n-1)}$ is the identity matrix. It follows from $e_{ij}(r)^{-1} = e_{ij}(-r)$ that T is a product of (upper triangular) elementary matrices. Similarly, any lower triangular T with 1s along the diagonal is a product of (lower triangular) elementary matrices.

Finally, we prove the claim that the commutator subgroup of $\mathrm{GL}_\infty(R)$ equals $\mathrm{E}_\infty(R)$. The commutation relation (7.5) and (7.3) show that each elementary matrix can be written as

$$e_{ik}(r) = e_{ij}(r) e_{jk}(1) e_{ij}(-r) e_{jk}(-1), \qquad k \neq i,$$

so it is a multiplicative commutator of elementary matrices. So

$$\mathrm{E}_\infty(R) \subseteq [\mathrm{E}_\infty(R), \mathrm{E}_\infty(R)]. \tag{7.6}$$

(In fact, they are equal.) Let $U, V \in \mathrm{GL}_\infty(R)$. By an abuse of notation, we write them as

$$\begin{pmatrix} U & 0_n \\ 0_n & 1_n \end{pmatrix}, \qquad \begin{pmatrix} V & 0_n \\ 0_n & 1_n \end{pmatrix}$$

for some suitable n. Then their commutator is

$$\begin{pmatrix} UVU^{-1}V^{-1} & 0_n \\ 0_n & 1_n \end{pmatrix} = \begin{pmatrix} U & 0_n \\ 0_n & U^{-1} \end{pmatrix} \begin{pmatrix} V & 0_n \\ 0_n & V^{-1} \end{pmatrix} \begin{pmatrix} U^{-1}V^{-1} & 0_n \\ 0_n & VU \end{pmatrix},$$

which is a product of elementary matrices by the first part of the Lemma. Since U, V were arbitrary, we learn that

$$[\mathrm{GL}_\infty(R), \mathrm{GL}_\infty(R)] \subseteq \mathrm{E}_\infty(R). \tag{7.7}$$

However, $\mathrm{E}_\infty(R) \subseteq \mathrm{GL}_\infty(R)$ implies that

$$[\mathrm{E}_\infty(R), \mathrm{E}_\infty(R)] \subseteq [\mathrm{GL}_\infty(R), \mathrm{GL}_\infty(R)]. \tag{7.8}$$

Combining (7.6), (7.7), (7.8), we must have

$$[\mathrm{GL}_\infty(R), \mathrm{GL}_\infty(R)] = \mathrm{E}_\infty(R).$$

□

The Whitehead Lemma also allows us to write the product in $K_1(R)$ "additively" as

$$\begin{aligned} [U] \cdot [V] = [UV] = \left[\begin{pmatrix} UV & 0 \\ 0 & 1 \end{pmatrix}\right] &= \left[\begin{pmatrix} UV & 0 \\ 0 & 1 \end{pmatrix}\right] \cdot \underbrace{\left[\begin{pmatrix} V^{-1} & 0 \\ 0 & V \end{pmatrix}\right]}_{\text{trivial}} \\ &= \left[\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}\right], \end{aligned} \tag{7.9}$$

where we abused notation and regarded U, V as some finite $n \times n$ matrix, and 1 denotes the $n \times n$ identity matrix.

7.3 Determinants for commutative rings

Let R be a unital *commutative* ring, then its group of units, $R^\times = \mathrm{GL}_1(R)$, is abelian. For each $n \geq 1$, there is a group homomorphism

$$\begin{aligned} \det_n : \mathrm{GL}_n(R) &\to R^\times, \qquad n \geq 1 \\ U &\mapsto \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot U_{1,\sigma(1)} \dots U_{n,\sigma(n)}, \end{aligned}$$

where S_n denotes the group of permutations on n objects. These determinants are compatible with the diagram of embeddings $\mathrm{GL}_m(R) \to \mathrm{GL}_n(R)$,

$$\det_n(\mathrm{diag}(U, 1_{n-m})) = \det_m(U), \qquad U \in \mathrm{GL}_m(R), \qquad \forall\, m \leq n.$$

So $\det_n$ maps give a cone from the embedding diagram to $R^\times$. By the universal property of the colimit $\mathrm{GL}_\infty(R)$, there is a well-defined group homomorphism

$$\det : \mathrm{GL}_\infty(R) \to R^\times,$$

which is such that

$$\det(\mathrm{diag}(U, 1_\infty)) = \det_n(U), \qquad \forall\, n \geq 1,\, U \in \mathrm{GL}_n(R).$$

Furthermore, since $R^\times$ is abelian, by the universal property (7.1) of abelianization, the above determinant map factorizes uniquely through $K_1(R) = \mathrm{GL}_\infty(R)_{\mathrm{ab}}$,

$$\begin{array}{ccc} \mathrm{GL}_\infty(R) & \xrightarrow{\det} & R^\times \\ {\scriptstyle \mathrm{mod}\ \mathrm{E}_\infty(R)}\downarrow & \nearrow_{\check{\det}} & \\ K_1(R) & & \end{array}, \qquad \check{\det}[U] = \det(U).$$

Informally, the determinant is "obviously" trivial on elementary matrices, and because $E_\infty(R)$ is precisely the commutator subgroup of $\mathrm{GL}_\infty(R)$, only the abelianized $\check{det}$ matters.

In the reverse direction to $\check{\det}$, there is also a homomorphism

$$\iota : R^\times = \mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}_\infty(R) \to K_1(R),$$

which satisfies

$$\check{\det} \circ \iota = \mathrm{id}_{R^\times}.$$

In particular, ι is injective.

To summarize, for commutative R, there is a split short exact sequence in **Ab**

$$1 \longrightarrow SK_1(R) \longrightarrow K_1(R) \underset{\iota}{\overset{\check{\det}}{\rightleftarrows}} R^\times \longrightarrow 1, \tag{7.10}$$

where

$$SK_1(R) := \ker \check{\det},$$

Remark. The 'S' in SK_1 denotes 'special', in analogy to the special linear groups with trivial determinant. Indeed, if we write $\mathrm{SL}_\infty(R) = \ker \det$, then $E_\infty(R)$ is also the commutator subgroup of $\mathrm{SL}_\infty(R)$, and $\mathrm{SL}_\infty(R)/E_\infty(R) \cong SK_1(R)$. (Exercise.)

Proposition 7.4. *If F is a field, then* $\check{\det} : K_1(F) \to F^\times$ *is an isomorphism.*

Proof. Let $U \in \mathrm{GL}_\infty(F)$, regarded as an invertible $n \times n$ matrix for some n. Then U can be turned into an invertible diagonal matrix via elementary row operations. This is an iterative process involving (i) swapping rows, and (ii) adding suitable multiples of one row to another. In Remark 4, we saw that row operations of type (ii) are implemented by multiplying with some elementary matrix. For type (i) row swapping operations, in the adjacent row case, we multiply by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = e_{12}(1)e_{21}(-1)e_{12}(1),$$

and similarly in the non-adjacent case. Note that we did not use $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as we wanted to swap rows while preserving the determinant.

Details: As U is invertible, its first column has some non-zero entry $\lambda_1 \in F^\times$. We swap rows to bring λ_1 to the $(1,1)$ position. Then add suitable multiples of $\lambda_1^{-1}\cdot$(first row) to each of the lower rows to bring U into the form

$$U \rightsquigarrow \begin{pmatrix} \lambda_1 & * \\ 0 & U^{(1)} \end{pmatrix}.$$

Repeat for the invertible $(n-1) \times (n-1)$ matrix $U^{(1)}$, and so on.

Thus, $E_1 U = T$ with $E_1 \in \mathrm{E}_n(F)$ and T an invertible upper triangular matrix. Using row operations of type (ii), we also obtain $E_2 T = D$ with $E_2 \in \mathrm{E}_n(F)$ and $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ an invertible diagonal matrix. Finally, we turn D into $\mathrm{diag}(\lambda, 1, \ldots, 1)$, by iteratively multiplying on the left by matrices of the form $\mathrm{diag}(1, \ldots, 1, \mu, \mu^{-1}, 1, \ldots, 1)$, which lie in $\mathrm{E}_n(F)$ because of Lemma 7.3.

Altogether, we have found that any $U \in \mathrm{GL}_\infty(F)$ satisfies $EU = \mathrm{diag}(\lambda, 1, \ldots)$ for some $E \in \mathrm{E}_\infty(F)$ and $\lambda \in F^\times$. Therefore, any $K_1(F)$ class has the form $[\mathrm{diag}(\lambda, 1, \ldots)]$ for some $\lambda \in F^\times$. This shows that $\iota : F^\times \to K_1(F)$ is surjective. By the split exact sequence (7.10), $\check{\det}$ gives an isomorphism from $K_1(F)$ to $F^\times$. □

Proposition 7.4 generalizes to Euclidean domains, but the proof requires an extra step involving Euclidean division algorithm.

Proposition 7.5. *If R is a Euclidean domain, then* $\check{\det} : K_1(R) \to R^\times$ *is an isomorphism. In particular,* $K_1(\mathbb{Z}) = \mathbb{Z}^\times = \{\pm 1\}$.

Proof. We write out the argument for $R = \mathbb{Z}$. (It is similar for Euclidean domains, only the notation is more complicated.)

Let $U \in \mathrm{GL}_\infty(\mathbb{Z})$, regarded as an $n \times n$ invertible matrix. The row-reduction algorithm of Prop. 7.4 required the nonzero λ_1 in the first column of U to have an inverse, but we cannot guarantee this unless λ_1 has minimal norm (i.e., $\lambda_1 = \pm 1$).

In general, the best we can do is to start from the nonzero λ_1 in the first column with the smallest norm. If λ_1 is not invertible, $\lambda_1\mathbb{Z}$ is a proper ideal in $\mathbb{Z}$. Since U is invertible, the dot product of the first row of U^{-1} with the first column of U must equal 1. This dot product lies in the ideal generated by the entries of the first column of U. As this ideal contains 1, it must be all of $\mathbb{Z}$. Therefore, there must be another nonzero λ_1' in the first column, with $\lambda_1' \notin \lambda_1\mathbb{Z}$.

We have $\lambda_1' = q\lambda_1 + r$ for some quotient q and some remainder r with $0 < |r| < |\lambda_1|$. By a type (ii) row operation, we can modify λ' into r, thus obtaining a non-zero entry in the first column with smaller norm than λ_1. Repeating this argument, we eventually end up with a non-zero entry μ_1 in the first column with norm 1, and this μ_1 is invertible. Now we can proceed as in Prop. 7.4 to convert U to the form $\begin{pmatrix} \mu_1 & * \\ 0 & U^{(1)} \end{pmatrix}$.

Repeat the same procedure for $U^{(1)}$ and so on, to get an upper-triangular matrix with invertible diagonal entries $\mu_i \in \{\pm 1\}$. The rest of the algorithm follows that of Prop. 7.4. We conclude that $EU = \mathrm{diag}(\mu, 1, \ldots, 1)$ for some suitable $E \in \mathrm{E}_n(F)$, with $\mu = \mu_1 \cdots \mu_n \in \mathbb{Z}^\times = \{\pm 1\}$. As in Prop. 7.4, we conclude that $\check{\det} : K_1(\mathbb{Z}) \to \mathbb{Z}^\times = \{\pm 1\}$ is an isomorphism. □

7.3.1 Determinant when entries are noncommutative?

For general *noncommutative* rings R, we still have the group homomorphism

$$\iota : R^\times = \mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}_\infty(R) \to K_1(R).$$

However, there is generally no determinant homomorphism from $K_1(R)$ to $R^\times$ because $R^\times$ is a non-abelian group.

We can still ask whether ι is surjective. This is interesting because if surjectivity fails, then there exist $K_1(R)$ classes whose representatives must come from $\mathrm{GL}_n(R)$ with $n > 1$.

Proposition 7.6. *If R is a division ring (e.g., the quaternions), then $\iota : R^\times \to K_1(R)$ is surjective.*

Proof. Any nonzero element of the division ring R is invertible. Then the algorithm in the proof of Prop. 7.4 shows that any $U \in \mathrm{GL}_\infty(R)$ has the form $EU = D$ for some $E \in \mathrm{E}_\infty(R)$ and some $D = \mathrm{diag}(\lambda, 1, \ldots), \lambda \in R^\times$. This means that

$$\iota : R^\times \to K_1(R), \qquad \lambda \mapsto [\mathrm{diag}(\lambda, 1, \ldots)]$$

is surjective. □

Since $K_1(R)$ is abelian, $\iota : R^\times \to K_1(R)$ factors through an abelian group homomorphism

$$\check{\iota} : R^\times_{\mathrm{ab}} \to K_1(R).$$

When ι is surjective (e.g., for division rings, as in Prop. 7.6), then $\check{\iota}$ is also surjective.

In fact, for *local rings*[12], surjectivity of ι (or $\check{\iota}$) holds. Furthermore, in this case, one can construct a "determinant" homomorphism $\det : \mathrm{GL}_\infty(R) \to R^\times_{\mathrm{ab}}$, inducing an abelianized "determinant" $\check{\det} : K_1(R) \to R^\times_{\mathrm{ab}}$, such that

$$\check{\det} \circ \check{\iota} = \mathrm{id}_{R^\times_{\mathrm{ab}}}$$

holds. So $\check{\iota}$ is also injective, thus $\check{\det} : K_1(R) \cong R^\times_{\mathrm{ab}}$ is an isomorphism. Details can be found on pp. 64–69 of Rosenberg's *Algebraic K-theory and its Applications*.

7.4 Whitehead group

Given a group G, the (integral) *group ring* is the ring $\mathbb{Z}G$ whose underlying abelian group is freely generated by the symbols $\delta_g, g \in G$. On generators, the multiplication rule is

$$\delta_g \star \delta_{g'} = \delta_{gg'}, \qquad g, g' \in G,$$

and this is extended $\mathbb{Z}$-bilinearly. The multiplicative identity of $\mathbb{Z}G$ is δ_e, where e is the identity element of G.

[12] Rings with a unique maximal left ideal; equivalently, a unique maximal right ideal.

It is not so easy to characterize what the invertible elements of $\mathbb{Z}G$ are, but there are certainly obvious ones like $\pm\delta_g, g \in G$. So there is an obvious subgroup of $K_1(\mathbb{Z}G)$, namely,

$$\{[\pm\delta_g] \,:\, g \in G\}.$$

(Here, $[\pm\delta_g]$ really means $[\mathrm{diag}(\pm\delta_g, \delta_e, \delta_e, \ldots)]$.)

The *Whitehead group* of G is defined to be

$$\mathrm{Wh}(G) := K_1(\mathbb{Z}G)/\{[\pm\delta_g] \,:\, g \in G\}.$$

As mentioned in Section 7.1, the Whitehead torsions detecting non-triviality of h-cobordisms from M_0 lie in $\mathrm{Wh}(\pi_1(M_0))$. Unfortunately, Whitehead groups and K_1 groups of $\mathbb{Z}G$ are difficult to compute.

Nevertheless, we have already computed the Whitehead group in one important case. By Prop. 7.5, we have $K_1(\mathbb{Z}) = \{\pm 1\}$, where the non-trivial element -1 is $[\mathrm{diag}(-\delta_e, \delta_e, \delta_e, \delta_e)] = [-\delta_e]$. Therefore, the Whitehead group of the trivial group $\{e\}$ is trivial,

$$\mathrm{Wh}(\{e\}) = K_1(\mathbb{Z}\{e\})/\{[\delta_e], [-\delta_e]\} = K_1(\mathbb{Z})/\{\pm 1\} = \{1\}.$$