11 Algebraic K_2

11.1 Steinberg group

Definition 12. Let R be a unital ring. For $n \geq 3$, the n-th *Steinberg group* $\mathrm{St}_n(R)$ is the free group generated by elements

$$x_{ij}(r), \qquad 1 \leq i \neq j \leq n,\ r \in R,$$

subject to the "commutation relations"

$$x_{ij}(r)x_{lk}(s) = x_{lk}(r)x_{ij}(s), \qquad i \neq k, j \neq l, \tag{11.1}$$
$$x_{ij}(r)x_{jk}(s) = x_{ik}(rs) \cdot x_{jk}(s)x_{ij}(r), \qquad i \neq k, \tag{11.2}$$
$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s). \tag{11.3}$$

Remark 8. There are no Steinberg relations relating

$$x_{ij}(r)x_{ji}(s), \qquad x_{ji}(s)x_{ij}(r).$$

Remark. Eq. (11.3) implies

$$x_{ij}(0) = 1, \qquad x_{ij}(-r) = x_{ij}(r)^{-1},$$

then (11.2) can be turned into the related commutation relation

$$x_{ji}(r)x_{kj}(s) = x_{ki}(-sr) \cdot x_{kj}(s)x_{ji}(r), \qquad i \neq k. \tag{11.2$'$}$$

Clearly, the group $\mathrm{E}_n(R)$ generated by $n \times n$ elementary matrices with entries in R, satisfies the Steinberg group relations (11.1), (11.2), (11.3). Therefore, there are surjective group homomorphisms

$$\begin{aligned} \mathrm{St}_n(R) &\to \mathrm{E}_n(R) \\ x_{ij}(r) &\mapsto e_{ij}(r). \end{aligned} \tag{11.4}$$

For $n \geq m \geq 3$, there are homomorphisms

$$\mathrm{St}_m(R) \to \mathrm{St}_n(R),$$

given by mapping generators to generators. However, beware that these homomorphisms may not be injective, since there are more relations in $\mathrm{St}_n(R)$ than in $\mathrm{St}_m(R)$. The *Steinberg group* is

$$\mathrm{St}(R) = \varinjlim \mathrm{St}_n(R),$$

and may be realized by letting $e_{ij}(r), 1 \leq i \neq j > \infty$, $r \in R$ be generators, subject to (11.1), (11.2), (11.3).

Note that the maps (11.4) are compatible with the diagrams $\mathrm{St}_m(R) \to \mathrm{St}_n(R)$ and $\mathrm{E}_m(R) \hookrightarrow \mathrm{E}_n(R)$, so there is a homomorphism

$$\varphi : \mathrm{St}(R) \twoheadrightarrow \mathrm{E}(R), \tag{11.5}$$

where we simplify notation and just write $\mathrm{E}(R)$ for $\mathrm{E}_\infty(R)$.

Suppose $\phi : R \to R'$ is a ring homomorphism. We can map the generators of $\mathrm{St}(R)$ to $\mathrm{St}(R')$, via

$$x_{ij}(r) \mapsto x_{ij}(\phi(r)).$$

Furthermore, this is compatible with the Steinberg group relations, particularly (11.2), (11.3). Therefore, we get an induced group homomorphism

$$\phi_* : \mathrm{St}(R) \to \mathrm{St}(R').$$

Similarly, there is an induced group homomorphism

$$\phi_* : \mathrm{E}(R) \to \mathrm{E}(R').$$

Definition 13. For a unital ring R, we define

$$K_2(R) := \ker\Big(\varphi : \mathrm{St}(R) \twoheadrightarrow \mathrm{E}(R)\Big)$$

Note that if $\phi : R \to R'$ is a ring homomorphism, then there is a commutative diagram of short exact sequences,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(R) & \longrightarrow & \mathrm{St}(R) & \xrightarrow{\varphi} & \mathrm{E}(R) & \longrightarrow & 1 \\ & & \vdots\,{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} & & \\ 1 & \longrightarrow & K_2(R') & \longrightarrow & \mathrm{St}(R') & \xrightarrow{\varphi} & \mathrm{E}(R') & \longrightarrow & 1 \end{array} . \tag{11.6}$$

So there is a well-defined homomorphism

$$\phi_* : K_2(R) \to K_2(R'),$$

given by restricting $\phi_* : \mathrm{St}(R) \to \mathrm{St}(R')$. Thus $K_2(-)$ is a functor from unital rings to groups. Shortly, we will show that it lands in *abelian* groups.

Remark. Suppose $K_2(R)$ vanishes, so $\mathrm{St}(R) \cong \mathrm{E}(R)$. This means that there are no further relations amongst elementary matrices apart from the "obvious" Steinberg relations. Thus, $K_2(R)$ can be interpreted as the (group generated by the) "non-obvious" relations enjoyed by concrete elementary matrices as compared to abstract Steinberg group generators.

Example 11.1. Let $u, v \in R^\times$ be commuting units. The Whitehead Lemma says that

$$d_{12}(u) := \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = e_{21}(u^{-1})e_{12}(1-u)e_{21}(-1)e_{12}(1-u^{-1})$$

is a product of four elementary matrices, thus it is an element of $\mathrm{E}_2(R)$; similarly for $d_{12}(v) := \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$. Because $d_{12}(u)$ and $d_{12}(v)$ are *diagonal matrices*, they commute, whatever the ring R is. As a relation, this is

$$d_{12}(u)d_{12}(v)d_{12}(u)^{-1}d_{12}(v)^{-1} = 1 \in \mathrm{E}(R). \tag{11.7}$$

However, the concept of "diagonal matrix" is external to the Steinberg relations. In other words, if we write out (11.7) as a product of elementary matrices using the Whitehead Lemma, we cannot deduce the triviality just from the abstract Steinberg relations (consider Remark 8).

Example 11.2. Next, consider

$$d_{12}(u) = \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad d_{13}(v) = \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{pmatrix}.$$

For commuting units $u, v \in R^\times$, concrete matrix multiplication gives

$$d_{12}(u)d_{13}(v)d_{12}(u)^{-1}d_{13}(v)^{-1} = 1 \in \mathrm{E}(R),$$

which is another example of a relation enjoyed by elementary matrices. If we write out the left side as a product of elementary matrices using the Whitehead Lemma, we can *occasionally* deduce the triviality just from the abstract Steinberg relations, depending on what u and v are.

In other words, the corresponding Steinberg group element,

$$D_{12}(u)D_{13}(v)D_{12}(u)^{-1}D_{13}(v)^{-1} \in \mathrm{St}(R), \tag{11.8}$$

where

$$\begin{aligned} D_{12}(u) &= s_{21}(u^{-1})s_{12}(1-u)s_{21}(-1)s_{12}(1-u^{-1}), \\ D_{13}(v) &= s_{31}(v^{-1})s_{13}(1-v)s_{31}(-1)s_{13}(1-v^{-1}), \end{aligned} \tag{11.9}$$

is generally not trivial! Note that (11.8) belongs to $K_2(R) \equiv \ker(\mathrm{St}(R) \twoheadrightarrow \mathrm{E}(R))$, since it is a lift of $1 \in \mathrm{E}(R)$.

Definition 14. Let $u, v \in R^\times$ be commuting units. Their *Steinberg symbol* is the element of $K_2(R)$ given by the expression (11.8), where $D_{12}(u), D_{13}(v)$ are any lifts of $d_{12}(u), d_{13}(v) \in \mathrm{E}(R)$ to $\mathrm{St}(R)$.

The flexibility to choose arbitrary lifts in Definition 14 will be useful, but it also means that the definition has some ambiguity at the moment. We proceed to resolve this issue.

11.2 Abelian K_2

In this subsection R denotes a unital ring, and $\varphi : \mathrm{St}(R) \twoheadrightarrow \mathrm{E}(R)$ is the surjective map whose kernel is $K_2(R)$.

Lemma 11.1. *The centre of* $\mathrm{E}(R)$ *is trivial.*

Proof. Suppose $U \in \mathrm{E}(R)$ commutes with all elementary matrices $e_{ij}(r), r \in R$. In particular, it commutes with $e_{ij}(1) - 1_\infty$. The latter is just the zero matrix with a single entry 1 in the ij-entry. Check that if U commutes with $e_{ij}(1) - 1_\infty$, then $U_{ii} = U_{jj}$ while other entries are zero. As $i \neq j$ are arbitrary, and the diagonal entries of U eventually become 1, this means that U is just the identity matrix. □

Lemma 11.2. *For each* $n \geq 3$*, the subgroup* $N_{i<j\leq n} \subseteq \mathrm{St}_n(R)$ *generated by Steinberg group generators* $x_{ij}(r), r \in R$ *with* $i < j \leq n$ *is isomorphic to the upper triangular subgroup of* $\mathrm{E}_n(R)$ *via the map* φ*.*

Proof. The additive abelian group underlying R^{n-1} is isomorphic to the multiplicative subgroup of "first-row" elementary matrices

$$\begin{pmatrix} 1 & r_2 & r_3 & \cdots & r_n \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 1 & 0 \\ 0 & & & & 1 \end{pmatrix}, \qquad (r_2, \ldots, r_n) \in R^{n-1}.$$

The above matrix is just a product

$$e_{12}(r_2)\cdots e_{1n}(r_n)$$

of commuting elementary matrices. Analogously, let $N_1 \subseteq N_{i<j\leq n} \subseteq \mathrm{St}_n(R)$ be the subgroup generated by the generators $x_{12}(r_2), \ldots, x_{1n}(r_n)$. Inspecting the Steinberg relations (11.1) and (11.3), we see that N_1 is abelian, and there are homomorphisms

$$\begin{array}{ccccc} R^{n-1} & \twoheadrightarrow & N_1 & \xrightarrow{\varphi} & \mathrm{E}_n(R) \\ (r_2,\ldots,r_n) & \mapsto & x_{12}(r_2)\cdots x_{1n}(r_n) & \mapsto & e_{12}(r_2)\cdots e_{1n}(r_n), \end{array}$$

whose composition is injective. Then φ must be injective on N_1.

Next, the subgroup of "first-two-rows" elementary matrices contains the "first-row" elementary matrices as a normal subgroup. The quotient is isomorphic to the "second-row" elementary matrices, which are identified with R^{n-2}. Let $N_2 \subseteq N_{i<j\leq n}$ be the subgroup generated by $x_{12}(-), \ldots, x_{1n}(-)$ and $x_{23}(-), \ldots, x_{2n}(-)$. A special case of the Steinberg commutation relation (11.2) is

$$x_{2j}(r)x_{12}(s)x_{2j}(r)^{-1} = x_{1j}(-sr), \qquad j = 3,\ldots,n,$$

showing that N_1 is likewise a normal subgroup in N_2. Then N_2/N_1 is isomorphic to the (abelian) subgroup generated by $x_{23}(-), \ldots, x_{2n}(-)$. In a similar fashion, there are homomorphisms

$$\begin{array}{ccccc} R^{n-2} & \twoheadrightarrow & N_2/N_1 & \xrightarrow{\varphi} & \mathrm{E}_n(R) \\ (r_3,\ldots,r_n) & \mapsto & x_{23}(r_3)\cdots x_{2n}(r_n) & \mapsto & e_{23}(r_3)\cdots e_{2n}(r_n), \end{array}$$

which implies that φ is injective on N_2/N_1. Combined with the earlier map $N_1 \to \mathrm{E}_n(R)$, we get an injective homomorphism $\varphi : N_2 \to \mathrm{E}_n(R)$.

Inductively, we get injective homomorphisms $N_{k+1}/N_k \to \mathrm{E}_n(R)$, which can be combined with $N_k \to \mathrm{E}_n(R)$ to get an injective $N_{k+1} \to \mathrm{E}_n(R)$ Continuing in this way, we obtain injectivity of the map $\varphi : N_{i<j\leq n} \to \mathrm{E}_n(R)$, which also surjects onto the upper triangular elementary matrices. □

Theorem 11.3. *$K_2(R)$ is the centre of $\mathrm{St}(R)$. In particular, $K_2(R)$ is abelian.*

Proof. [$Z(\mathrm{St}(R)) \subseteq K_2(R)$.] Let x be in the centre of $\mathrm{St}(R)$. Then $\varphi(x)$ commutes with $\varphi(\mathrm{St}(R)) = \mathrm{E}(R)$ (as φ is surjective). But Lemma 11.1 says that $\mathrm{E}(R)$ has trivial centre. So $\varphi(x) = 1$, thus $x \in \ker\varphi = K_2(R)$.

[$K_2(R) \subseteq Z(\mathrm{St}(R))$.] Let $x = x_{i_1 j_1}(r_1) \cdots x_{i_m j_m}(r_m) \in K_2(R)$, thus

$$\varphi(x) = e_{i_1 j_1}(r_1) \cdots e_{i_m j_m}(r_m) = 1 \in \mathrm{E}(R).$$

Pick any $n > i_1, \ldots, i_m, j_1, \ldots, j_m$, and write A_n for the "last-column" subgroup of $\mathrm{St}(R)$ generated by $x_{1n}(-), \ldots, x_{n-1,n}(-)$. Steinberg relations say that

$$x_{i_l j_l}(r_l) x_{kn}(s) x_{i_l j_l}(r_l)^{-1} = \begin{cases} x_{kn}(s), & j_l \neq k, \\ x_{i_l n}(r_l s) x_{kn}(s), & j_l = k. \end{cases}, \qquad l = 1, \ldots, m; k = 1, \ldots, n-1.$$

This means that x normalizes A_n.

So for arbitrary $y \in A_n$, we have $xyx^{-1}y^{-1} \in A_n$. At the same time, $\varphi(x) = 1$ means that

$$\varphi(xyx^{-1}y^{-1}) = \varphi(x)\varphi(y)\varphi(x)^{-1}\varphi(y)^{-1} = 1.$$

But Lemma 11.2 says that φ is injective on $N_{i<j\leq n} \supset A_n$, thus $xyx^{-1}y^{-1} = 1$. In other words, x commutes with all elements of A_n. In particular, x commutes any $x_{an}(-)$ with $a < n$.

We may run a similar analysis for "lower triangular" Steinberg symbols, and deduce that x commutes with "last-row" generators $x_{nb}(-)$ with $b < n$. Then, since

$$x_{ab}(-) = x_{an}(-) x_{nb}(1) x_{an}(-)^{-1} x_{nb}(1)^{-1}, \qquad a, b < n,$$

and n can be arbitrarily large, we have shown that x commutes with the entire $\mathrm{St}(R)$. □

Theorem 11.3 ensures that Definition 14 of Steinberg symbols is well-defined. Because different choices of lifts $D_{12}(u), D_{13}(v)$ of $d_{12}(u), d_{13}(v)$ will only differ by $\ker \varphi = K_2(R)$ terms, which we now know are central in $\mathrm{St}(R)$, there is no ambiguity in the formula $\{u, v\} = D_{12}(u) D_{13}(v) D_{12}(u)^{-1} D_{13}(v)^{-1}$.

11.3 Connecting K_2 and K_1

11.3.1 Relative K_2

For an ideal I in a unital ring R, the *augmented ring* is

$$R \oplus I, \qquad (r_1, s_1)(r_2, s_2) := (r_1 r_2, r_1 s_2 + r_2 s_1 + s_1 s_2). \tag{11.10}$$

(Compare with the unitization $\mathbb{Z} \oplus I$.) There is a natural "addition" ring homomorphism

$$\begin{aligned} \alpha : R \oplus I &\to R \\ (r, s) &\mapsto r + s, \end{aligned} \tag{11.11}$$

and this induces a group homomorphism $\mathrm{St}(R \oplus I) \to \mathrm{St}(R)$. We restrict this to the normal subgroup

$$\widetilde{\mathrm{St}}(R, I) := \mathrm{ncl}\{x_{ij}(0, s) \,:\, s \in I\} \subseteq \mathrm{St}(R \oplus I),$$

whose generators are conjugates of I-Steinberg symbols. Explicitly,

$$\begin{aligned} \alpha_* : \widetilde{\mathrm{St}}(R, I) &\to \mathrm{St}(R) \\ x_{kl}(r, t)x_{ij}(0, s)x_{kl}(r, t)^{-1} &\mapsto x_{kl}(r + t)x_{ij}(s)x_{kl}(r + t)^{-1}. \end{aligned} \tag{11.12}$$

Composing α_* with $\varphi : \mathrm{St}(R) \to \mathrm{E}(R)$, we see that

$$\begin{aligned} \mathrm{Range}\Big(\varphi \circ \alpha_* : \widetilde{\mathrm{St}}(R, I) \to \mathrm{GL}(R)\Big) &= \mathrm{ncl}_{\mathrm{E}(R)}\{e_{ij}(s) \,:\, s \in I\} \\ &= \mathrm{ncl}_{\mathrm{GL}(R)}\{e_{ij}(s) \,:\, s \in I\} \equiv \mathrm{E}(R, I). \end{aligned} \tag{11.13}$$

It is tempting to define $K_2(R, I)$ as the kernel of $\varphi \circ \alpha_* : \widetilde{\mathrm{St}}(R, I) \to \mathrm{E}(R, I)$, mimicking the absolute case. However, the resulting group will be too large, and perhaps non-abelian. For example, α_* automatically makes all commutators of the form

$$\underbrace{x_{kl}(t, -t)x_{ij}(0, s)x_{kl}(t, -t)^{-1}}_{\text{generic generator of } \widetilde{\mathrm{St}}(R,I)} x_{ij}(0, s)^{-1}, \qquad s, t \in I,$$

trivial in the Steinberg group $\mathrm{St}(R)$, even before applying φ. So we should actually define α_* on the quotient

$$\mathrm{St}(R, I) := \widetilde{\mathrm{St}}(R, I)/\mathrm{ncl}\{x_{kl}(t, -t)x_{ij}(0, s)x_{kl}(t, -t)^{-1}x_{ij}(0, s)^{-1} \,:\, s, t \in I\}, \tag{11.14}$$

and regard $\varphi \circ \alpha_*$ as the map

$$\begin{array}{ccccc} & & \mathrm{St}(R, I) & \xrightarrow{\alpha_*} \mathrm{St}(R) & \xrightarrow{\varphi} \mathrm{E}(R, I) \subseteq \mathrm{GL}(R, I) \subseteq \mathrm{GL}(R) \\ \big[x_{kl}(r, t)x_{ij}(0, s)x_{kl}(r, t)^{-1}\big] & \mapsto & & e_{kl}(r + t)e_{ij}(s)e_{kl}(r + t)^{-1}. & \end{array}$$

Definition 15 (Weibel's book pp. 222, Keune '78)**.** For an ideal I in a unital ring R, the *relative K_2-group* is

$$K_2(R,I) := \ker\big(\varphi\circ\alpha_* : \mathrm{St}(R,I)\to \mathrm{E}(R,I)\big).$$

Remark. We have $\iota : R \to R\oplus I$. Consider the map

$$\begin{aligned}\iota_*\circ\alpha_* : \mathrm{St}(R,I) &\to \mathrm{St}(R\oplus I)\\ \big[x_{kl}(r,t)x_{ij}(0,s)x_{kl}(r,t)\big] &\mapsto x_{kl}(r+t,0)x_{ij}(s,0)x_{kl}(r+t,0),\end{aligned}$$

which is an injective homomorphism (exercise). If $x\in K_2(R,I)\subseteq \mathrm{St}(R,I)$, then $\varphi\circ\alpha_*(x)=1$, thus

$$1=\iota_*\circ\varphi\circ\alpha_*(x)=\varphi_{R\oplus I}\circ\iota_*\circ\alpha_*(x).$$

So $\iota_*\circ\alpha_*(K_2(R,I))$ is a subgroup of the abelian group $K_2(R\oplus I)$, and we deduce that $K_2(R,I)$ is abelian.

Proposition 11.4. *There is an exact sequence*

$$\mathrm{St}(R,I)\xrightarrow{\alpha_*}\mathrm{St}(R)\xrightarrow{q_*}\mathrm{St}(R/I)\to 1.$$

Proof.

- [Exactness at $\mathrm{St}(R/I)$.] Each generator $x_{ij}(\dot r)\in\mathrm{St}(R/I)$ ($\dot r\in R/I$) is obtained as $q_*(x_{ij}(r))=x_{ij}(\dot r)$ by taking any $r\in R$ such that $q(r)=\dot r$. Thus q_* is surjective.

- Next, we show that $H := \mathrm{Range}\,\alpha_* = \ker q_*$.

  - By formula (11.12),

    $$H=\mathrm{Range}\Big(\alpha_*:\mathrm{St}(R,I)\to\mathrm{St}(R)\Big)=\mathrm{ncl}_{\mathrm{St}(R)}\{x_{ij}(s)\ :\ s\in I\}. \tag{11.15}$$

    Any generator $x_{kl}(r)x_{ij}(s)x_{kl}(r)^{-1}, r\in R, s\in I$ of the right side is killed by q_*. So $H\subseteq\ker q_*$.

  - Note that H is a normal subgroup of $\mathrm{St}(R)$, and we saw that $H\subseteq\ker q_*$. So the map $q_*:\mathrm{St}(R)\to\mathrm{St}(R/I)$ descends to a homomorphism

    $$\begin{aligned}\check q : \mathrm{St}(R)/H &\to \mathrm{St}(R/I)\\ (x_{i_1j_i}(r_1)\cdots x_{i_mj_m}(r_m))\cdot H &\mapsto x_{i_1j_i}(r_1+I)\cdots x_{i_mj_m}(r_m+I).\end{aligned}$$

The quotient group $\mathrm{St}(R)/H$ has generators $x_{ij}(r)\cdot H$, $r\in R$, but some of these are repeated. For example,

$$x_{ij}(r+s)\cdot H = x_{ij}(r)\underbrace{x_{ij}(s)}_{\in H}\cdot H = x_{ij}(r)\cdot H, \qquad s\in I.$$

So we may unambiguously write

$$y_{ij}(r+I) := x_{ij}(r)\cdot H, \qquad r+I\in R/I,\ \ 1\le i\neq j<\infty,$$

and use these $y_{ij}(r+I)$ as a set of generators for $\mathrm{St}(R)/H$.
Let us check that $y_{ij}(r+I)$ satisfy the Steinberg relations (for the ring R/I). For example,

$$\begin{aligned} y_{ij}(r_1+I)y_{lk}(r_2+I) &= \big(x_{ij}(r_1)\cdot H\big)\big(x_{lk}(r_2)\cdot H\big)\\ &= \big(x_{ij}(r_1)x_{lk}(r_2)\big)\cdot H\\ &= \big(x_{lk}(r_2)x_{ij}(r_1)\big)\cdot H\\ &= \big(x_{lk}(r_2)\cdot H\big)\big(x_{ij}(r_1)\cdot H\big)\\ &= y_{lk}(r_2+I)y_{ij}(r_1+I), \qquad i\neq k, j\neq l. \end{aligned}$$

Similarly for the other Steinberg relations. Therefore, there is a well-defined homomorphism

$$\begin{aligned} \phi : \mathrm{St}(R/I) &\to \mathrm{St}(R)/H\\ x_{ij}(r+I) &\mapsto y_{ij}(r+I) = x_{ij}(r)\cdot H. \end{aligned}$$

Recalling the formula for $\check{q} : \mathrm{St}(R)/H \to \mathrm{St}(R/I)$, we have found an inverse ϕ to $\check{q}$. Thus $\mathrm{St}(R)/H$ is isomorphic to $\mathrm{St}(R/I)$.
But we also have $\mathrm{St}(R/I)$ isomorphic to $\mathrm{St}(R)/\ker q_*$ (as q_* is surjective). So $H=\ker q_*$ as claimed.

□

11.3.2 Connecting map

Let R be a unital ring and I an ideal. Our constructions of (relative) K_1 and K_2 groups lead to the following commuting diagram of groups with exact rows

(adapted from Rosenberg's textbook pp. 200, and Weibel's K-book pp. 223):

$$\begin{array}{ccccccccccc}
 & & & & & & 1 & & & & \\
 & & & & & & \downarrow & & & & \\
1 & \longrightarrow & K_2(R,I) & \longrightarrow & \mathrm{St}(R,I) & \xrightarrow{\varphi\circ\alpha_*} & \mathrm{GL}(R,I) & \xrightarrow{\mathrm{mod\,E}(R,I)} & K_1(R,I) & \longrightarrow & 1 \\
 & & \vdots\,{\scriptstyle\alpha_*} & & \downarrow{\scriptstyle\alpha_*} & & \downarrow & & \downarrow{\scriptstyle j} & & \\
1 & \longrightarrow & K_2(R) & \longrightarrow & \mathrm{St}(R) & \xrightarrow{\varphi_R} & \mathrm{GL}(R) & \xrightarrow{\mathrm{mod\,E}(R)} & K_1(R) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \\
1 & \longrightarrow & K_2(R/I) & \longrightarrow & \mathrm{St}(R/I) & \xrightarrow{\varphi_{R/I}} & \mathrm{GL}(R/I) & \xrightarrow{\mathrm{mod\,E}(R/I)} & K_1(R/I) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 1 & & 1 & & & &
\end{array} \tag{11.16}$$

Exactness of the second and third rows is by definitions of the absolute K_1 and K_2 groups (which are abelian by extra work). Exactness of the first row is by construction (also see (11.13)).

A notational remark: we wrote $\varphi_R : \mathrm{St}(R) \to \mathrm{GL}(R)$ for the map $\varphi : \mathrm{St}(R) \twoheadrightarrow \mathrm{E}(R)$ with enlarged codomain $\mathrm{GL}(R)$; similarly for $\varphi_{R/I}$. A few other comments:

- Recalling the definition of j in (10.6), the top right square commutes. The top-middle square commutes by construction.
- Recalling (11.6), the quotient ring homomorphism $q : R \to R/I$ induces the group homomorphisms

  $$\mathrm{GL}(R) \to \mathrm{GL}(R/I), \quad \mathrm{St}(R) \to \mathrm{St}(R/I), \quad \mathrm{GL}(R) \to \mathrm{GL}(R/I)$$

  (all denoted q_*), as well as the abelianized group homomorphism $q_* : K_1(R) \to K_1(R/I)$, such that all the lower squares commute.
- The $\mathrm{GL}(-)$ column is exact by definition.
- The column of K_1 groups is exact by Prop. 10.2.
- The column of $\mathrm{St}(-)$ groups is exact by Prop. 11.4.
- The map $\alpha_* : K_2(R,I) \to K_2(R)$ is defined to be the restriction of $\alpha_* : \mathrm{St}(R,I) \to \mathrm{St}(R)$. Here, $\alpha_* : K_2(R,I) \xrightarrow{\alpha} K_2(R)$ lands within $K_2(R)$

by a diagram chase with the φ maps. Exactness of the $K_2(-)$ column is likewise inherited from exactness of the $\mathrm{St}(-)$ column. (Exercise)

The following theorem joins the K_1 and K_2 half-exact sequences.

Theorem 11.5. *There is a canonical connecting homomorphism $\partial : K_2(R/I) \to K_1(R, I)$ such that*

$$K_2(R, I) \xrightarrow{\alpha_*} K_2(R) \xrightarrow{q_*} K_2(R/I) \xrightarrow{\partial} K_1(R, I) \xrightarrow{j} K_1(R) \xrightarrow{q_*} K_1(R/I)$$

is exact.

Proof. Having established the commutative diagram (11.16) with exact rows/columns (particularly the extra work to establish Prop. 11.4, the theorem follows from the snake lemma (but see Section 11.3.3). It is instructive, though, to directly spell out what ∂ is, why it is well-defined, and check exactness at $K_2(R/I)$ and at $K_1(R, I)$.

- [Definition of ∂.] Let $x \in K_2(R/I) = \ker(\varphi_{R/I}) \subseteq \mathrm{St}(R/I)$. As q_* is surjective, pick $y \in \mathrm{St}(R)$ such that $q_*(y) = x$. Then the lower-middle commutative square of (11.16) gives

  $$q_* \circ \varphi_R(y) = \varphi_{R/I} \circ q_*(y) = \varphi_{R/I}(x) = 1.$$

  Thus

  $$\varphi_R(y) \in \ker(q_*) = \mathrm{GL}(R, I), \tag{11.17}$$

  and we define

  $$\partial(x) := [\varphi_R(y)]_{K_1(R,I)}. \tag{11.18}$$

- [Well-definedness of ∂.] Suppose we used a different lift y' of x to define $\partial(x)$. Then $y'y^{-1} \in \ker q_* = \mathrm{Range}\, \alpha_*$ by exactness of St column. Pick any $z \in \mathrm{St}(R, I)$ satisfying $\alpha_*(z) = y'y^{-1}$, so

  $$\varphi_R(y') = \varphi_R(y'y^{-1})\varphi_R(y) = \big(\varphi \circ \alpha_*(z)\big)\varphi_R(y).$$

  But $\varphi \circ \alpha_*(\tilde{y}) \in \mathrm{Range}\, \varphi \circ \alpha_* = \mathrm{E}(R, I)$ by exactness of the first row. Thus

  $$[\varphi_R(y')]_{K_1(R,I)} = [\varphi_R(y)]_{K_1(R)}.$$

- [∂ is a homomorphism.] This is similar to the check for well-definedness.

- [Exactness at $K_2(R/I)$.] For any $y \in K_2(R)$, we have $\varphi_R(y) = 1$ by definition, thus
$$\partial \circ q_*(y) = [\varphi_R(y)]_{K_1(R,I)} = 1.$$
So Range $q_* \subseteq \ker \partial$.

  Conversely, suppose $x \in \ker \partial$. Following the definition (11.18), this means that
$$1 = \partial(x) = [\varphi_R(y)]_{K_1(R,I)}, \qquad y \in \mathrm{St}(R) \,:\, q_*(y) = x. \tag{11.19}$$
Eq. (11.19) says that $\varphi_R(y) \in \mathrm{E}(R,I)$, and the latter equals Range$(\varphi \circ \alpha_*)$ by first row exactness. So, pick $a \in \mathrm{St}(R,I)$ satisfying $\varphi \circ \alpha_*(a) = \varphi_R(y) = \varphi(y)$. So $\varphi_R(\alpha_*(a^{-1})y) = 1$. By exactness of the second row, $\alpha_*(a^{-1})y \in K_2(R)$. Simultaneously, we also know that
$$q_*(\alpha_*(a^{-1})y) = \underbrace{q_*(\alpha_*(a^{-1}))}_{=1} q_*(y) = q_*(y) = x.$$
by St-column exactness. So $x \in$ Range q_*.

- [Exactness at $K_1(R,I)$. For any $x \in K_2(R/I)$,
$$j \circ \partial(x) = j[\varphi_R(y)]_{K_1(R,I)}, \qquad y \in \mathrm{St}(R) \,:\, q_*(y) = x.$$
When defining ∂, we saw that $\varphi_R(y) \in \mathrm{GL}(R,I)$. By the top right commutative diagram in (11.16),
$$j \circ \partial(x) = j[\varphi_R(y)]_{K_1(R)} = j(\varphi(y)) \bmod \mathrm{E}(R,I) = \varphi_R(y) \bmod \mathrm{E}(R) = 1,$$
where the last equality is due to second row exactness in (11.16). This shows that Range $\partial \subseteq \ker j$.

  Conversely, let $[u]_{K_1(R,I)} \in \ker j$, represented by some $u \in \mathrm{GL}(R,I)$. So $j[u]_{K_1(R,I)} \equiv [u]_{K_1(R)} = 1$. By second row exactness, there exists some $y \in \mathrm{St}(R)$ with $\varphi_R(y) = u$. Set $x = q_*(y) \in \mathrm{St}(R/I)$. Then
$$\varphi_{R/I}(x) = \varphi_{R/I}(q_*(y)) = q_*(\varphi_R(y)) = q_*(u) = 1,$$
where the last equality is due to $u \in \mathrm{GL}(R,I) = \ker q_*$. So $x \in \ker \varphi_{R/I} = K_2(R/I)$. Furthermore, by definition of ∂,
$$\partial(x) = [\varphi_R(y)]_{K_1(R,I)} = [u]_{K_1(R,I)}.$$
This shows that $\ker j \subseteq$ Range ∂.

□

11.3.3 Role of snake lemma?

The snake lemma is usually presented in the setting of *abelian groups* (or R-modules, or *abelian* categories), starting from a commutative diagram

$$\begin{array}{ccc} & & 1 \\ & & \downarrow \\ G_1 & \longrightarrow & H_1 \\ \downarrow & & \downarrow \\ G_2 & \longrightarrow & H_2 \\ \downarrow & & \downarrow \\ G_3 & \longrightarrow & H_3 \\ \downarrow & & \\ 1 & & \end{array} \tag{11.20}$$

with exact columns. (The consensus seems to be to present (11.20) in a reflected manner, with a G-row above a H-row.) In this abelian setting, kernels and cokernels for the horizontal maps automatically exist and allow one to obtain a big diagram like (11.16). Then "diagram-chasing" leads to the connecting map ∂ joining the kernel column to the cokernel column in an exact manner. This is typically applied to short exact sequences of chain complexes (coming from algebraic topology). However, there are conflicting slogans on the "correct categories" for snake lemmas:

> *An abelian category is a category satisfying just enough axioms so the snake lemma holds* — Stacks project.

> *... category theory could not grasp either the conceptual foundations of ... the Snake Lemma, which remain valid and strongly meaningful in the category* **Grp** *of groups, even if this category does not belong to the abelian setting in which these lemmas are generally proved in a significant categorical way* — Borceux, Bourn: Mal'cev, Protomodular, Homological, and Semi-Abelian Categories.

In our setting, we are working with *non-abelian* groups. If there is a big diagram of groups like (11.16), where the inclusions are of *normal* subgroups

(so they are kernels of the next maps), then the diagram chase works as in the abelian case. However, it is not true that any diagram of groups like (11.20) automatically leads to a big diagram with kernel and cokernel columns like (11.16). We could do this in the case of K_2 (kernel) and K_1 (cokernel) after substantial work, involving special properties of Steinberg groups and elementary matrices.

More importantly, the definition of $\mathrm{St}(R, I)$, which corresponds to G_1 in (11.20), is not obvious. It appears in such a way that an exact St column is available, as well as an exact first row with kernels and cokernels. Subsequently, diagram-chasing works as in the abelian case.

In any case, our primary concern is not snake lemmas themselves, but setting up connecting maps specific to K-theory. It would be misleading to directly attribute $\partial : K_2(R/I) \to K_1(R, I)$ to general "abstract nonsense". To illustrate this point, suppose we did not bother to construct $\mathrm{St}(R, I)$, thus we do not have G_1 in (11.20) and the top left part of (11.16). We can still define the connecting map $\partial : K_2(R/I) \to K_1(R, I)$ in the same way. The diagram-chasing proof for the unambiguity of ∂ and exactness at $K_2(R/I)$ is not available, but they are still true, because of special properties of the $\mathrm{St}(-)$ and $\mathrm{E}(-)$ groups. This latter "non-snake-lemma" approach is detailed in Rosenberg's textbook.

Another important comment regarding snake lemma: the choice of G_1 making the following diagram (11.21) exact and commute,

$$\begin{array}{ccc} & & 1 \\ & & \downarrow \\ G_1 & \longrightarrow & \mathrm{GL}(R, I) \\ \downarrow & & \downarrow \\ \mathrm{St}(R) & \longrightarrow & \mathrm{GL}(R) \\ \downarrow & & \downarrow \\ \mathrm{St}(R/I) & \longrightarrow & \mathrm{GL}(R/I) \\ \downarrow & & \\ 1 & & \end{array} \tag{11.21}$$

is not unique. Provided $G_1 \to \mathrm{GL}(R, I)$ has a cokernel, we would get an exact

connecting map

$$K_2(R) \to K_2(R/I) \equiv \ker(\mathrm{St}(R/I) \to \mathrm{GL}(R/I)) \xrightarrow{\partial} \mathrm{coker}(G_1 \to \mathrm{GL}(R,I)) \to K_1(R) \to \cdots,$$

but we specifically want $\mathrm{coker}(G_1 \to \mathrm{GL}(R,I))$ to be $K_1(R,I)$. But even the latter requirement does not uniquely specify G_1, and therefore

$$\ker(G_1 \to \mathrm{GL}(R/I)) \text{ “} = K_2(R,I)\text{”}$$

still has some ambiguity.

Here is a guiding principle for deciding upon the “correct” G_1 to use. It should result in an *abelian* $K_2(R,I) = \ker(G_1 \to \mathrm{GL}(R/I))$ which is able to receive, in an exact manner, a K_3-to-K_2 connecting map from some abelian group $K_3(R/I)$, where K_3 is some functor $\mathbf{Ring} \to \mathbf{Ab}$. As it turns out, the choice $G_1 = \mathrm{St}(R,I)$ of Definition 15 works. (More precisely, unlike other easier choices, it does not violate this principle.) In turn, we are prompted to look for the next functor $K_3 : \mathbf{Ring} \to \mathbf{Ab}$.

Contrast the situation of K-theory with that of ordinary (co)homology groups defined via chain complexes. The (abelian) chain groups are available in all degrees $n \geq 0$, and the analogue of (11.20) is available for every $n \geq 0$. In K-theory, we do not have such a uniform “abelian categorical” construction, so the work needed to set up the functors $K_n(-)$, as well as the connecting maps between them, is rather different.

11.4 Non-trivial K_2 elements via Steinberg symbols

11.4.1 Some properties of Steinberg symbols

Let $A, B \in \mathrm{E}(R)$ be *commuting* matrices. We want to understand whether their commutativity is due solely to abstract Steinberg relations, or whether it is due to some special properties of concrete matrix algebra. So consider

$$A \star B := \tilde{A}\tilde{B}\tilde{A}^{-1}\tilde{B}^{-1} \ \in K_2(R), \tag{11.22}$$

Note that

$$\varphi(A \star B) = ABA^{-1}B^{-1} = 1, \qquad (A, B \text{ commute})$$

so $A \star B$ is indeed an element of $K_2(R)$, and it is possibly non-trivial. Theorem 11.3 ensures that (11.22) is well-defined, independently of the choice of lifts.

Here are some properties of $\star$.

- Suppose $S \in \mathrm{GL}(R)$. Work with some matrix size n such that $A, B \in \mathrm{E}_n(R)$ and $S \in \mathrm{GL}_n(R)$. Doubling matrix sizes,

$$SAS^{-1} \sim \begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & S \end{pmatrix},$$

and similarly for SBS^{-1} regarded as a $2n \times 2n$ matrix. By the Whitehead Lemma, $\mathrm{diag}(S, S^{-1}) \in \mathrm{E}_{2n}(R)$, so it admits a lift $\tilde{S} \in \mathrm{St}(R)$. Then (working in $2n \times 2n$),

$$\varphi(\tilde{S}\tilde{A}\tilde{S}^{-1}) = SAS^{-1}, \qquad \varphi(\tilde{S}\tilde{B}\tilde{S}^{-1}) = SBS^{-1}.$$

Therefore,

$$\begin{aligned} SAS^{-1} \star SBS^{-1} &= (\tilde{S}\tilde{A}\tilde{S}^{-1})(\tilde{S}\tilde{B}\tilde{S}^{-1})(\tilde{S}\tilde{A}\tilde{S}^{-1})^{-1}(\tilde{S}\tilde{B}\tilde{S}^{-1})^{-1} \\ &= \tilde{S}(\tilde{A}\tilde{B}\tilde{A}^{-1}\tilde{B}^{-1})\tilde{S}^{-1} \\ &= \tilde{S}(\underbrace{A \star B}_{\text{central}})\tilde{S}^{-1} \\ &= A \star B. \end{aligned}$$

Thus $\star$ is conjugation invariant.

- A brief calculation (exercise) shows that $\star$ is skew-symmetric and bilinear,

$$\begin{aligned} (A \star B)(B \star A) &= 1, \\ (A_1 A_2) \star B &= (A_1 \star B)(A_2 \star B). \end{aligned}$$

The Steinberg symbol for commuting units $u, v \in R^\times$ can be expressed as

$$\{u, v\} = d_{12}(u) \star d_{13}(v) \in K_2(R),$$

from which conjugation invariance and bilinearity follows. We also have

$$\{1, u\} = d_{12}(1) \star d_{12}(u) = 1 = \{u, 1\}, \qquad \forall\, u \in R^\times. \tag{11.23}$$

Exercise 11.1. For $i \neq j$ and $u \in R^\times$, let $d_{ij}(u)$ be the diagonal matrix with u as the i-th diagonal entry, u^{-1} as the j-th diagonal entry, and 1 for the remaining diagonal entries. Show that for commuting units $u, v \in R^\times$,

$$\{u, v\} = d_{ij}(u) \star d_{ik}(v)$$

for any choice of distinct i, j, k. Hence, or otherwise, deduce that $\{u, v\} = \{v, u\}^{-1}$.

11.4.2 Determinant of commutator revisited

Let us revisit Section 10.2 with our K_2-tools at hand. We have a relative determinant homomorphism $D : \mathrm{GL}(R, I) \to G$. Let $U, V \in \mathrm{GL}_1(R)$, and suppose $UVU^{-1}V^{-1}$ is D-class. We want to understand what controls $D(UVU^{-1}V^{-1})$. We saw that it depends on $q(U), q(V)$ in some way, where $q : R \to R/I$ is the quotient map.

Recall that $UVU^{1-}V^{-1}$ being D-class implies that $q(U), q(V)$ commute in $\mathrm{GL}_1(R/I)$, so there is a Steinberg symbol

$$\{q(U), q(V)\} = D_{12}(q(U))D_{13}(q(V))D_{12}(q(U))^{-1}D_{13}(q(V))^{-1} \in K_2(R/I) \subseteq \mathrm{St}(R/I),$$

where $D_{12}(-), D_{13}(-)$ are given by (11.8). Let us see what ∂ does to it. According to the defining equation (11.18), we first pick a lift $y \in \mathrm{St}(R)$ of $\{q(U), q(V)\}$. For example, since $D_{12}(q(U))$ and $D_{13}(q(V))$ are certain products of Steinberg group generators (for $\mathrm{St}(R/I)$) we can choose

$$y = D_{12}(U)D_{13}(V)D_{12}(U)^{-1}D_{13}(V)^{-1}$$

to be the same product of Steinberg group generators for $\mathrm{St}(R)$. Then

$$\varphi_R(y) = d_{12}(U)d_{13}(V)d_{12}(U)^{-1}d_{13}(V)^{-1} \in \mathrm{GL}(R, I),$$

since φ_R just turns each $x_{ij}(r)$ into $e_{ij}(r)$. So

$$\begin{aligned}
\partial\{q(U), q(V)\} &= \left[d_{12}(U)d_{13}(V)d_{12}(U)^{-1}d_{13}(V)^{-1}\right]_{K_1(R,I)} \\
&= \left[\begin{pmatrix} UVU^{-1}V^{-1} & 0 & 0 \\ 0 & U^{-1}1U1 & 0 \\ 0 & 0 & 1V^{-1}1V \end{pmatrix}\right]_{K_1(R,I)} \\
&= [UVU^{-1}V^{-1}]_{K_1(R,I)}.
\end{aligned}$$

Thus $\{q(U), q(V)\}$ is the precursor of $[UVU^{-1}V^{-1}]_{K_1(R,I)}$, as anticipated in Section 10.2. We have

$$D(UVU^{-1}V^{-1}) = \check{D}[UVU^{-1}V^{-1}]_{K_1(R,I)} = \check{D} \circ \partial\{q(U), q(V)\}. \qquad (11.24)$$

Remark. For D the Fredholm determinant, the relation (11.24) to Steinberg symbols was observed by L. Brown in 1975.

Eq. (11.24) shows that in order for $D(UVU^{-1}V^{-1})$ to be non-trivial, the Steinberg symbol $\{q(U), q(V)\}$ must be non-trivial in the first place. This means that the commutativity of $q(U), q(V)$ is not due to some "obvious" reason (such as (10.10)–(10.11)), but is due to some deeper-lying matrix relations encoded by $\{q(U), q(V)\} \in K_2(R/I)$.

By the way: the existence of non-trivial $\det(UVU^{-1}V^{-1})$ examples shows that the groups

$$K_2(\mathcal{L}/\mathcal{L}^1), \qquad K_1(\mathcal{L}, \mathcal{L}^1)$$

are non-trivial. Indeed, the part of $K_2(\mathcal{L}/\mathcal{L}^1)$ generated by Steinberg symbols is already non-trivial, admitting a surjective homomorphism onto $\mathbb{C}^\times$.

11.5 Spin

Physicists are extremely familiar with the *Pauli spin matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The spin Lie group in three dimensions is

$$\mathrm{Spin}(3) \cong \mathrm{SU}(2) = \left\{ \begin{pmatrix} w & -\bar{z} \\ z & \bar{w} \end{pmatrix} \,:\, |w|^2 + |z|^2 = 1 \right\},$$

together with the double-covering map

$$\begin{aligned} \phi : \mathrm{Spin}(3) &\to \mathrm{SO}(3) \\ U &\mapsto R_{ij} = \frac{1}{2}\mathrm{Tr}(\sigma_i U \sigma_j U^*). \end{aligned}$$

Note that ϕ is a group homomorphism, and we have a short exact sequence of groups,

$$1 \to \{\pm 1\} \to \mathrm{Spin}(3) \xrightarrow{\phi} \mathrm{SO}(3) \to 1.$$

It is well-known that in Euclidean 3-space, generic rotations about different axes do not commute (SO(3) is non-abelian). Nevertheless, the following SO(3) matrices commute.

$$d_{12}(-1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad d_{13}(-1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

They represent rotations by angle π around the z-axis and the y-axis, respectively. We do *not* expect this commutativity to be attributable to Steinberg relations enjoyed by SO(3) matrices (which are $\mathrm{E}_3(\mathbb{R})$ elements, see below). This is because a generic change in the axes/rotation angle ruins the commutativity, whereas the Steinberg relations remain unaffected.

The following are lifts of $d_{12}(-1)$ and $d_{13}(-1)$ in Spin(3),

$$\hat{d}_{12}(-1) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overset{\phi}{\mapsto} d_{12}(-1), \qquad \hat{d}_{13}(-1) := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \overset{\phi}{\mapsto} d_{13}(-1).$$

Informally, $\hat{d}_{12}(-1), \hat{d}_{13}(-1) \in \mathrm{Spin}(3)$ are "rotations in spinor space $\mathbb{C}^2$" by angle π. You may have heard about the infamous property of spinorial rotations, that a 4π rotation is required in order to return to the identity transformation. Whereas an 2π-rotation in SO(3) is already equal to the identity transformation. We can see a related "weird" property of spinorial rotations, by noticing that the commutator of $\hat{d}_{12}(-1)$ and $\hat{d}_{13}(-1)$ is no longer trivial,

$$\hat{d}_{12}(-1)\hat{d}_{13}(-1)\hat{d}_{12}(-1)^{-1}\hat{d}_{13}(-1)^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{Spin}(3), \tag{11.25}$$

in fact being equal to a 2π-spinorial rotation (about any "axis").

To proceed further and relate the above "anomalous commutator" to Steinberg symbols, we borrow some results coming from topology of Lie groups (not examinable). It is well-known that $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$ when $n \geq 3$, and the inclusions $\mathrm{SO}(3) \to \mathrm{SO}(n)$ are isomorphisms on the fundamental groups $\pi_1(-)$. Note that the universal (double) cover of SO(3) is Spin(3). By a polar decomposition argument, $\mathrm{SL}(n, \mathbb{R})$ deformation retracts to $\mathrm{SO}(n)$, so the inclusions $\mathrm{SO}(3) \to \mathrm{SL}(3, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$ induce isomorphisms on π_1. Then for the double covers $\widetilde{\mathrm{SL}(n, \mathbb{R})}$ of $\mathrm{SL}(2, \mathbb{R})$, we have inclusions $\mathrm{Spin}(3) \to \widetilde{\mathrm{SL}(3, \mathbb{R})} \to \widetilde{\mathrm{SL}(n, \mathbb{R})}$.

With these preparations, we can now relate (11.25) to a Steinberg symbol. As $\mathbb{R}$ is a field, recall from the proof of Prop. 7.4 that $\mathrm{SL}(n, \mathbb{R})$ coincides with $\mathrm{E}_n(\mathbb{R})$. So we have the double covers $\widetilde{\mathrm{SL}(n, \mathbb{R})} \to \mathrm{E}_n(\mathbb{R})$. For our purposes, it is more important that we have a compatible family of *central extensions*,

$$1 \to \{\pm 1\} \to \widetilde{\mathrm{SL}(n, \mathbb{R})} \to \mathrm{E}_n(\mathbb{R}) \to 1, \qquad n \geq 3.$$

Passing to the colimits, we have a central extension

$$1 \to \{\pm 1\} \to \varinjlim \widetilde{\mathrm{SL}(n, \mathbb{R})} \to \mathrm{E}(\mathbb{R}) \to 1,$$

A key abstract characterization of $\varphi : \mathrm{St}(R) \twoheadrightarrow \mathrm{E}(R)$, is that it is the *universal central extension* of the group $\mathrm{E}(R)$. While we did not formally define or prove this, what it implies is that there is a unique homomorphism $f : \mathrm{St}(\mathbb{R}) \to \varinjlim \widetilde{\mathrm{SL}(n,\mathbb{R})}$ leading to a commuting diagram of central extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(\mathbb{R}) & \longrightarrow & \mathrm{St}(\mathbb{R}) & \xrightarrow{\varphi} & \mathrm{E}(\mathbb{R}) & \longrightarrow & 1 \\ & & \vdots & & \downarrow f & & \downarrow = & & \\ 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \varinjlim \widetilde{\mathrm{SL}(n,\mathbb{R})} & \longrightarrow & \mathrm{E}(\mathbb{R}) & \longrightarrow & 1 \end{array} .$$

Now, $-1 \in \mathbb{R}^\times$, and we may consider the Steinberg symbol $\{-1,-1\} \in K_2(\mathbb{R})$. By definition,

$$\{-1,-1\} = D_{12}(-1)D_{13}(-1)D_{12}(-1)^{-1}D_{13}(-1)^{-1},$$

where $\varphi(D_{12}(-1)) = d_{12}(-1)$ and $\varphi(D_{13}(-1)) = d_{13}(-1)$ in $\mathrm{E}(\mathbb{R})$. We know that $d_{12}(-1)$ lifts to $\pm\hat{d}_{12}(-1) \in \mathrm{Spin}(3)$ (included inside $\widetilde{\mathrm{SL}(n,\mathbb{R})}$), and likewise, $d_{13}(-1)$ lifts to $\pm\hat{d}_{13}(-1)$. This means that

$$f\{-1,-1\} = \hat{d}_{12}(-1)\hat{d}_{13}(-1)\hat{d}_{12}(-1)^{-1}\hat{d}_{13}(-1)^{-1} \quad \text{included into } \varinjlim \widetilde{\mathrm{SL}(n,\mathbb{R})},$$

which is the 2-torsion element found in (11.25). This shows that $\{-1,-1\} \in K_2(\mathbb{R})$ is a non-trivial Steinberg symbol.

11.6 Steinberg symbol for ring of integers

Quite generally, for any unital ring R, we may consider the Steinberg group element

$$y = x_{12}(1)x_{21}(-1)x_{12}(1) \in \mathrm{St}(R).$$

The image in $\mathrm{E}(R)$ is

$$\varphi_R(y) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It follows that

$$(\varphi_R(y))^4 = \varphi_R(y^4) = 1 \;\Rightarrow\; y^4 \in K_2(R).$$

Let us investigate

$$x := y^4 = \big(x_{12}(1)x_{21}(-1)x_{12}(1)\big)^4 \tag{11.26}$$

in the case $R = \mathbb{Z}$. Because K_2 is a functor, under the homomorphism $\iota : \mathbb{Z} \to \mathbb{R}$, we also have $\iota_*(x) \in K_2(\mathbb{R}) \subseteq \mathrm{St}(\mathbb{R})$, as well as

$$\varphi_{\mathbb{R}}(\iota_*(y)) = \iota_*(\varphi_{\mathbb{Z}}(y)) = \iota_* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{E}(\mathbb{R}). \tag{11.27}$$

The last matrix is a rotation about the z-axis by angle $-\pi/2$. If you are familiar with spin groups, the Spin(3) element lifting this matrix is

$$\pm \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \in \mathrm{Spin}(3) \hookrightarrow \varinjlim \widetilde{\mathrm{SL}(n, \mathbb{R})}.$$

Following the element (11.27) around the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K_2(\mathbb{Z}) & \longrightarrow & \mathrm{St}(\mathbb{Z}) & \xrightarrow{\varphi_{\mathbb{Z}}} & \mathrm{E}(\mathbb{Z}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \\
1 & \longrightarrow & K_2(\mathbb{R}) & \longrightarrow & \mathrm{St}(\mathbb{R}) & \xrightarrow{\varphi_{\mathbb{R}}} & \mathrm{E}(\mathbb{R}) & \longrightarrow & 1 \\
 & & \vdots & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle =} & & \\
1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \varinjlim \widetilde{\mathrm{SL}(n, \mathbb{R})} & \xrightarrow{2:1} & \mathrm{E}(\mathbb{R}) & \longrightarrow & 1
\end{array},$$

we have

$$f(\iota_*(y)) = \pm \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}.$$

Thus,

$$f(\iota_*(x)) = f(\iota_*(y))^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This shows that $x \in K_2(\mathbb{Z})$ is non-trivial.

Now, $\mathbb{Z}^\times = \{\pm 1\}$, so that by (11.23), the only possibly non-trivial Steinberg symbol for $\mathbb{Z}$ is $\{-1, -1\}$. In fact, a little algebra reveals that $x \in K_2(\mathbb{Z})$ of (11.26) is precisely $\{-1, -1\} \in K_2(\mathbb{Z})$. Furthermore, Milnor was able to calculate $K_2(\mathbb{Z}) \cong \mathbb{Z}_2$ in his book *Introduction to Algebraic K-theory* (1971).

Finally, we mention that if we work with the ring $\mathbb{C}$, then the Steinberg symbol $\{-1, -1\} \in K_2(\mathbb{C})$ becomes trivial (Exercise).