14 Central extensions and homology of groups

14.1 Universal central extensions

A *central extension* of a group G by an (abelian) group A is a short exact sequence of groups

$$1 \to A \to E \xrightarrow{\varphi} G \to 1,$$

with A central.

If we fix G, but do not specify A, the data of a central extension is usually abbreviated to (E, φ). Then a morphism $(E, \varphi) \to (E', \varphi')$ is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \cdots\cdots\rightarrow & \ker\varphi & \cdots\cdots\rightarrow & E & \xrightarrow{\varphi} & G & \cdots\cdots\rightarrow & 1 \\ & & \vdots & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle =} & & \\ 1 & \cdots\cdots\rightarrow & \ker\varphi' & \cdots\cdots\rightarrow & E' & \xrightarrow{\varphi'} & G & \cdots\cdots\rightarrow & 1 \end{array} .$$

(The dotted arrows are available automatically, and do not constitute extra data.) We get a category of central extensions of G in this way.

If (E, φ) is isomorphic to some $(G \times A, \pi_G)$ (with A abelian, of course), then (E, φ) is said to be a *trivial central extension* (of G).

Definition 24. A central extension (E, φ) of G is *universal*, if for any other central extension (E', φ') of G, there exists a unique morphism $(E, \varphi) \to (E', \varphi')$.

Due to the universal property characterization, a universal central extension is unique up to isomorphism. The existence of a universal central extension is guaranteed under the following situation:

Theorem 14.1. *A group G admits a universal central extension iff G is* perfect[28]*. In this case, a central extension (E, φ) is universal iff*

- *E is perfect;*
- *E has no non-trivial central extensions.*

Proof. Suppose G is not perfect, so $G_{\mathrm{ab}} = G/[G, G]$ is non-trivial. Suppose, for a contradiction, that G admits a universal central extension (E, φ). Certainly,

[28]Meaning $G = [G, G]$.

$(G \times G_{\mathrm{ab}}, \pi_G)$ is a central extension of G. But we can construct distinct morphisms

$$\begin{aligned} \varphi \times 1_{G_{\mathrm{ab}}} &: (E, \varphi) \to (G \times G_{\mathrm{ab}}, \pi_G) \\ \varphi \times ((\mathrm{mod}\,[G,G]) \circ \varphi) &: (E, \varphi) \to (G \times G_{\mathrm{ab}}, \pi_G). \end{aligned}$$

This contradicts the universality of (E, φ).

The more difficult converse involves showing that any central extension satisfying the two conditions in the theorem is universal, then constructing such a central extensions using a presentation of the perfect group G. We refer to Rosenberg's book for details.) □

Let R be a unital ring. In Eq. (7.6), we learned that $\mathrm{E}(R)$ is a perfect group, so it admits a universal central extension by Theorem 14.1. In fact, the same argument shows that any Steinberg group generator $x_{ij}(r)$ is a commutator, e.g.,

$$x_{ij}(r) = x_{ik}(1)x_{kj}(r)x_{ik}(-1)x_{kj}(-r), \qquad k \neq i, j. \tag{14.1}$$

So $\mathrm{St}(R) \subseteq [\mathrm{St}(R), \mathrm{St}(R)] \subseteq \mathrm{St}(R)$, and equality holds throughout, i.e., $\mathrm{St}(R)$ is perfect. Therefore, by the second part of Theorem 14.1, we can demonstrate that $(\mathrm{St}(R), \varphi_R)$ is a universal central extension of $\mathrm{E}(R)$ if we can show that every central extension of $\mathrm{St}(R)$ is trivial.

Lemma 14.2. *For any unital ring R, every central extension of $\mathrm{St}(R)$ is trivial.*

Proof. Let $(\widetilde{\mathrm{St}(R)}, \psi)$ be a central extension of $\mathrm{St}(R)$. Our task is to construct a splitting homomorphism $s : \mathrm{St}(R) \to \widetilde{\mathrm{St}(R)}$ for ψ. It involves the following operation,

$$\begin{aligned} \mathrm{St}(R) \times \mathrm{St}(R) &\to \widetilde{\mathrm{St}(R)} \\ (x, y) &\mapsto x \star y := \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}, \end{aligned}$$

where $\widetilde{(-)}$ denotes any lift of $(-)$. As $\ker \psi$ is central, $x \star y$ is independent of the choices of lifts. It is clear that $y \star x = (x \star y)^{-1}$.

The formula for the splitting map is

$$\begin{aligned} s : \mathrm{St}(R) &\to \widetilde{\mathrm{St}(R)} \\ x_{ij}(r) &\mapsto s_{ij}(r), \end{aligned} \tag{14.2}$$

where

$$s_{ij}(r) := x_{ik}(1) \star x_{kj}(r), \qquad \text{any } k \neq i, j. \tag{14.3}$$

Note that

$$\psi(s_{ij}(r)) = [x_{ik}(1), x_{kj}(r)] = x_{ij}(r)$$

holds by the Steinberg relations. So $\psi \circ s = \mathrm{id}_{\mathrm{St}(R)}$ holds when applied to Steinberg group generators. The independence of (14.3) from the choice of k, will also need to be checked, and is due to (14.5) later.

To check that the formula (14.2) extends to a group homomorphism (thus a splitting *homomorphism*), some preparation is needed.

- We claim that

$$x_{ij}(r) \star x_{kl}(s) = 1, \qquad r, s \in R,\ i \neq l,\ j \neq k. \tag{14.4}$$

  To show this, pick an index m distinct from i, j, k, l, and pick lifts

$$a = \widetilde{x_{im}(1)}, \qquad b = \widetilde{x_{mj}(r)}, \qquad c = \widetilde{x_{kl}(s)}.$$

  By the identity (14.1), $aba^{-1}b^{-1}$ is a lift of $x_{ij}(r)$. Also, inspecting indices, $x_{kl}(s)$ commutes with $x_{im}(1)$ and with $x_{mj}(r)$. So $z = aca^{-1}c^{-1}$ and $z' = bcb^{-1}c^{-1}$ are lifts of 1, thus central, and

$$ac = zca, \qquad bc = z'cb.$$

  Then (14.4) is verified by the calculation

$$\begin{aligned} x_{ij}(r) \star x_{kl}(s) &= (aba^{-1}b^{-1})c(aba^{-1}b^{-1})^{-1}c^{-1} \\ &= aba^{-1}b^{-1}cbab^{-1}a^{-1}c^{-1} \\ &= (z')^{-1}aba^{-1}b^{-1}bcab^{-1}c^{-1}a^{-1}z \\ &= (z')^{-1}aba^{-1}z^{-1}acc^{-1}b^{-1}z')a^{-1}z \\ &= 1. \end{aligned}$$

- Next, we show that

$$x_{mj}(r) \star x_{jk}(s) = x_{mi}(1) \star x_{ik}(rs), \qquad r, s \in R,\ m, i, j, k \text{ distinct.} \tag{14.5}$$

  For this, pick lifts

$$u = \widetilde{x_{mi}(1)}, \qquad v = \widetilde{x_{ij}(r)}, \qquad w = \widetilde{x_{jk}(s)}.$$

We shall temporarily write $[\cdot,\cdot]$ for the *multiplicative* commutator. By Steinberg relations,

$$\begin{aligned} [u,v] = uvu^{-1}v^{-1} \quad &\text{lifts} \quad x_{mj}(r) \\ [v,w] = vwv^{-1}w^{-1} \quad &\text{lifts} \quad x_{ik}(rs). \end{aligned} \tag{14.6}$$

Eq. (14.4) gives

$$[u,w] = uwu^{-1}w^{-1} = x_{mi}(1) \star x_{jk}(s) = 1, \tag{14.7}$$

while Eq. (14.4) combined with (14.6) gives

$$\begin{aligned} [[u,v],u] &= x_{mj}(r) \star x_{mi}(1) = 1, \\ [[u,v],v] &= x_{mj}(r) \star x_{ij}(r) = 1, \\ [[u,v],[v,w]] &= x_{mj}(r) \star x_{ik}(rs) = 1. \end{aligned} \tag{14.8}$$

From the "$= 1$" identities (14.7)–(14.8), the subgroup G generated by u, v, w has the property that $[G,G]$ is abelian. The Jacobi identity (exercise) then comes into play,

$$[u,[v,w]][v,[w,u]][w,[u,v]] = 1.$$

Because of $[u,w] = 1$ (Eq. (14.7)), the Jacobi identity becomes

$$1 = [u,[v,w]][w,[u,v]] = \big(x_{mi}(1) \star x_{ik}(rs)\big)\big(x_{jk}(s) \star x_{mj}(r)\big) \quad ((14.6)),$$

which is equivalent to the claim (14.5) by antisymmetry of $\star$.

Now we check that the $s_{ij}(r)$ satisfy the Steinberg relations. The relations that we need to verify are

$$\begin{aligned} s_{ij}(r)s_{kl}(s) &= s_{kl}(s)s_{ij}(r) && i \neq l, j \neq k, \\ s_{ij}(r)s_{jk}(s) &= s_{ik}(rs)s_{jk}(s)s_{ij}(r), && i \neq k, \\ s_{ij}(r)s_{ij}(s) &= s_{ij}(r+s). \end{aligned}$$

- For the first relation, since $s_{ij}(r)$ lifts $x_{ij}(r)$ and $s_{kl}(s)$ lifts $x_{kl}(s)$, we have

$$s_{ij}(r)s_{kl}(s)s_{ij}(r)^{-1}s_{kl}(s)^{-1} = x_{ij}(r) \star x_{kl}(s) = 1. \qquad \text{(by (14.4))}$$

- Similarly, for the second relation,

$$\begin{aligned} s_{ij}(r)s_{jk}(s)s_{ij}(r)^{-1}s_{jk}(s)^{-1} &= x_{ij}(r) \star x_{jk}(s) \\ &= x_{ij}(1) \star x_{jk}(rs), \qquad \text{(by (14.5))} \\ &= s_{ik}(rs). \qquad \text{((14.3))} \end{aligned}$$

- For the third relation, let $k \neq i, j$, and pick

$$u = \widetilde{x_{ik}(1)}, \qquad v = \widetilde{x_{kj}(r)}, \qquad w = \widetilde{x_{kj}(s)},$$

so that according to definition (14.3), we have

$$s_{ij}(r)s_{ij}(s) = [u, v][u, w].$$

We verify that (exercise)

$$[u, v][u, w] = [u, vw][v, [w, u]].$$

But

$$[w, u] \quad \text{lifts} \quad x_{kj}(s)x_{ik}(1)x_{kj}(s)^{-1}x_{ik}(1)^{-1} = x_{ij}(-s),$$

so

$$[v, [w, u]] = x_{kj}(r) \star x_{ij}(-s) = 1. \qquad \text{(by (14.4) and } k \neq j)$$

Thus,

$$s_{ij}(r)s_{ij}(s) = [u, vw] = x_{ik}(1) \star x_{kj}(r + s) = s_{ij}(r + s).$$

□

Corollary 14.3. *For each unital ring R, the central extension*

$$1 \longrightarrow K_2(R) \longrightarrow \mathrm{St}(R) \xrightarrow{\varphi_R} \mathrm{E}(R) \longrightarrow 1$$

is a universal central extension of $\mathrm{E}(R)$.

Now that we know that $K_2(R) = \ker \varphi_R$ is the kernel of a universal central extension of $\mathrm{E}(R)$, we will be able to relate it to a certain (co)homology of the group $\mathrm{E}(R)$. The concept of (co)homology of groups will be provided in Section 14.2, but we state our desired result first, as motivation.

Theorem 14.4. *Let G be a perfect group, and (E, φ) be a universal central extension of G (which exists by Theorem 14.1.) Then* $\ker \varphi$ *is naturally isomorphic to $H_2(G, \mathbb{Z})$. In particular, for a unital ring R, we have*

$$K_2(R) \cong H_2(\mathrm{E}(R); \mathbb{Z}).$$

The proof of Theorem 14.4 involves quite a lot of theory of the homology groups $H_\bullet(G; M)$ beyond Section 14.2, and we merely refer to Chapter 4.1 of Rosenberg's book.

14.2 Homology of groups

Convention: (Following Rosenberg's book.) For a group G, the group ring is denoted $\mathbb{Z}[G]$. A $\mathbb{Z}[G]$-module can be viewed as an abelian group on which G acts by automorphisms. The *trivial* $\mathbb{Z}[G]$-module refers to the abelian group $\mathbb{Z}$ on which all elements of G act as $\mathrm{id}_{\mathbb{Z}}$. Both left and right $\mathbb{Z}[G]$-module actions will be considered.

For each $j \geq 0$, consider the free abelian group $\mathbb{Z}G^{j+1}$ with generating set G^{j+1}. It has the left and right $\mathbb{Z}[G]$-module actions

$$\begin{aligned} g \cdot (g_0, \ldots, g_j) &= (gg_0, \ldots, gg_j) \\ (g_0, \ldots, g_j) \cdot g &= (g_0 g, \ldots, g_j g). \end{aligned}$$

There are boundary maps

$$\begin{aligned} &d_j : \mathbb{Z}G^{j+1} \to \mathbb{Z}G^j \\ &(g_0, \ldots, g_j) \mapsto \sum_{k=0}^{j} (-1)^k (g_0, \ldots, \underbrace{\widehat{g_k}}_{\text{omit}}, \ldots, g_j), \qquad j \geq 1, \end{aligned}$$

which are $\mathbb{Z}[G]$-module homomorphisms. We also write $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$, $g \mapsto 1$. Then the sequence

$$\cdots \xrightarrow{d_{j+1}} \mathbb{Z}G^{j+1} \xrightarrow{d_j} \mathbb{Z}G^j \xrightarrow{d_{j-1}} \cdots \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \tag{14.9}$$

is a chain complex which is exact, i.e., all homology groups are zero (exercise).

We mention that $\mathbb{Z}G^{j+1}$ can be viewed as as a free $\mathbb{Z}[G]$-module with generators $(1, g_1, \ldots, g_j)$. Then (14.9) is a so-called *free resolution* of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$.

14.2.1 Homology of G

To get interesting homology groups, we tensor each $\mathbb{Z}G^{j+1}$ by some left $\mathbb{Z}[G]$-module M,

$$\mathbb{Z}G^{j+1} \otimes_{\mathbb{Z}[G]} M,$$

and consider the twisted boundary maps

$$d_{j,M} := d_j \otimes \mathrm{id}_M : \mathbb{Z}G^{j+1} \otimes_{\mathbb{Z}[G]} M \to \mathbb{Z}G^{j} \otimes_{\mathbb{Z}[G]} M, \qquad j \geq 1.$$

For $j = 0$, take $d_{0,M} : \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} M = M \to 0$ to be the zero map. We obtain a chain complex of left $\mathbb{Z}[G]$-modules, which might not be exact (even for the case of trivial coefficients $M = \mathbb{Z}$). Subsequently, we just write d_j instead of $d_{j,M}$.

The *homology groups of G with coefficients in M* are

$$H_j(G; M) := \ker d_j / \mathrm{Im}\, d_{j+1}, \qquad j \geq 0.$$

It is sometimes convenient to describe $H_j(G; M)$ using the following bijections

$$\begin{aligned} \mathbb{Z}G^{j+1} \otimes_{\mathbb{Z}[G]} M &\leftrightarrow \mathbb{Z}G^{j} \otimes_{\mathbb{Z}} M \\ (g_0, \ldots, g_j) \otimes m &\mapsto (g_1 g_0^{-1}, \ldots, g_j g_0^{-1}) \otimes g_0 m. \end{aligned}$$

Under this identification, the formula for the boundary maps become

$$d_j : \mathbb{Z}G^{j} \otimes_{\mathbb{Z}} M \to \mathbb{Z}G^{j-1} \otimes_{\mathbb{Z}} M$$

$$(g_1, \ldots, g_j) \otimes m \mapsto (g_2 g_1^{-1}, \ldots, g_j g_1^{-1}) \otimes g_1 m + \sum_{k=1}^{j} (-1)^k (g_1, \ldots, \widehat{g_k}, \ldots, g_j) \otimes m,$$

and $d_0 = 0 : M \to 0$. The formulae for the $j = 1$ and $j = 2$ cases are

$$\begin{aligned} d_1 : \mathbb{Z}G \otimes_{\mathbb{Z}} M &\to M & d_2 : \mathbb{Z}G^2 \otimes_{\mathbb{Z}} M &\to \mathbb{Z}G \otimes_{\mathbb{Z}} M \\ g_1 \otimes m &\mapsto g_1 m - m & (g_1, g_2) \otimes m &\mapsto g_2 g_1^{-1} \otimes g_1 m - g_2 \otimes m + g_1 \otimes m. \end{aligned}$$

Example 14.1. If the G-action on M is trivial, then $d_1 = 0$, thus

$$H_0(G; M) = M. \qquad \text{(trivial } G\text{-action)}$$

Also, $d_2 : (g_1, g_2) \otimes m \mapsto (g_2 g_1^{-1} - g_2 + g_1) \otimes m$. Then we may construct a group isomorphism (Exercise)

$$H_1(G; M) = \big(\mathbb{Z}G \otimes_{\mathbb{Z}} M\big) / \mathrm{Im}\, d_2 \to G_{\mathrm{ab}} \otimes_{\mathbb{Z}} M. \qquad \text{(trivial } G\text{-action)}$$

14.2.2 Cohomology of G

Fix a $\mathbb{Z}[G]$-module M. Dualizing d_j gives the coboundary maps

$$d^{j-1} : \mathrm{Hom}_{\mathbb{Z}[G]\text{-mod}}(\mathbb{Z}G^j, M) \to \mathrm{Hom}_{\mathbb{Z}[G]\text{-mod}}(\mathbb{Z}G^{j+1}, M).$$

The *cohomology groups of G with coefficients in M* are defined as

$$H^n(G; M) := \ker d^j / \mathrm{Im}\, d^{j-1}.$$

An alternative picture uses the identification

$$\mathrm{Hom}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}G^j; M) \cong \mathrm{Hom}_{\mathbb{Z}[G]\text{-mod}}(\mathbb{Z}G^{j+1}, M)$$

given by

$$\begin{aligned} f(g_1, \ldots, g_j) &= F(1, g_1, g_1g_2, \ldots, g_1 \cdots g_j) \\ F(g_0, \ldots, g_j) &= g_0 \cdot f(g_0^{-1}g_1, \ldots, g_{j-1}^{-1}g_j) \end{aligned}$$

Under this identification, the coboundary maps become

$$d^{j-1} : \mathrm{Hom}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}G^{j-1}, M) \to \mathrm{Hom}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}G^j, M)$$

$$\begin{aligned} (d^{j-1}f)(g_1, \ldots, g_j) = g_1 \cdot f(g_2, \ldots, g_j) &+ \sum_{k=1}^{j-1} (-1)^k f(g_1, \ldots, g_k g_{k+1}, \ldots, g_j) \\ &+ (-1)^j f(g_1, \ldots, g_{j-1}). \end{aligned}$$

Example 14.2. Suppose M has the trivial G-action. Then $d^0 = 0$, while for $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, M)$, we have the formula

$$(d^1 f)(g_1, g_2) = f(g_2) - f(g_1g_2) + f(g_1), \qquad g_1, g_2 \in G. \tag{14.10}$$

Then $H^1(G; M) = \ker d^1$ comprises the group homomorphisms $G \to M$.

Example 14.3. Similarly, for M with trivial G-action, the effect of d^2 on cochains $\sigma \in \mathrm{Hom}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}G^2, M)$ is

$$(d^2\sigma)(g_1, g_2, g_3) = \sigma(g_2, g_3) - \sigma(g_1g_2, g_3) + \sigma(g_1, g_2g_3) - \sigma(g_1, g_2). \tag{14.11}$$

Let us explain how 2-cocycles $\sigma \in \ker d^2$ arise from central extensions

$$1 \to M \to E \xrightarrow{\varphi} G \to 1. \tag{14.12}$$

of G by M. Strictly speaking, there is an extra piece of data here, namely, an identification $\iota : M \xrightarrow{\cong} \ker \varphi$.

Pick a set-theoretic section $s : G \to E$ such that $\varphi \circ s = \mathrm{id}_G$, normalized to satisfy $s(1_G) = s(1_E)$. In general, this s is not a homomorphism, and might even be obstructed from being one. The failure of s to be a homomorphism is measured by the function

$$\begin{aligned} \sigma : G^2 &\to M \\ \sigma(g_1, g_2) &= \iota^{-1}\big(s(g_1 g_2)s(g_2)^{-1}s(g_1)^{-1}\big). \end{aligned}$$

Now consider $g_1, g_2, g_3 \in G$. We have

$$\begin{aligned} s(g_1g_2g_3) = s((g_1g_2)g_3) &= \iota \circ \sigma(g_1g_2, g_3)s(g_1g_2)s(g_3) \\ &= \iota \circ \sigma(g_1g_2, g_3)\iota \circ \sigma(g_1, g_2)s(g_1)s(g_2)s(g_3), \end{aligned}$$

but due to associativity, this equals

$$\begin{aligned} s(g_1g_2g_3) = s(g_1(g_2g_3)) &= \iota \circ \sigma(g_1, g_2g_3)s(g_1)s(g_2g_3) \\ &= \iota \circ \sigma(g_1, g_2g_3)\iota \circ \sigma(g_2, g_3)s(g_1)s(g_2)s(g_3). \end{aligned}$$

Comparing the two expressions, and reverting to additive notation in M, we have

$$0 = \sigma(g_1g_2, g_3) + \sigma(g_1, g_2) - \sigma(g_1, g_2g_3) - \sigma(g_2, g_3) \overset{(14.11)}{=} (d^2\sigma)(g_1, g_2, g_3),$$

which says that σ is a 2-cocycle.

Suppose we used a different (normalized) section s', so there is some $f : G \to M$ such that

$$s'(g) = s(g)\iota \circ f(g), \qquad g \in G.$$

From s', we would obtain a 2-cocycle σ', such that

$$\begin{aligned} \iota \circ \sigma'(g_1, g_2) &= s'(g_1g_2)s'(g_2)^{-1}s'(g_1)^{-1} \\ &= s(g_1g_2)\iota \circ f(g_1g_2)\iota \circ f(g_2)^{-1}s(g_2)^{-1}\iota \circ f(g_1)^{-1}s(g_1)^{-1} \\ &= \iota \circ \sigma(g_1, g_2)\iota \circ f(g_1g_2)\iota \circ f(g_2)^{-1}\iota \circ f(g_1)^{-1}. \end{aligned}$$

Undoing ι and reverting to additive notation,

$$\sigma'(g_1g_2) - \sigma(g_1g_2) = f(g_1g_2) - f(g_1) - f(g_2) \overset{(14.10)}{=} -(d^1f)(g_1, g_2).$$

We conclude that each central extension (14.12) is associated with a well-defined *cohomology class* $\sigma \in H^2(G; M)$, where M has the trivial G-action.

Interestingly, the abelian group structure of $H^2(G;M)$ appears in the theory of central extensions of G by M as well. We say that (E,φ,ι) is a *parametrized central extension* of G by M if (E,φ) is a central extension of G and $\iota : M \to \ker\varphi$ is an isomorphism. We say that (E,φ,ι) and (E',φ',ι') are *isomorphic*, if there is a commutative diagram of short exact sequences,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \xrightarrow{\iota} & E & \xrightarrow{\varphi} & G & \longrightarrow & 1 \\ & & \downarrow{=} & & \downarrow{\cong} & & \downarrow{=} & & \\ 1 & \longrightarrow & M & \xrightarrow{\iota'} & E' & \xrightarrow{\varphi'} & G & \longrightarrow & 1 \end{array}$$

The set of isomorphism classes of *parametrized central extension* of G by M is denoted

$$\mathrm{Ext}(G,M).$$

On $\mathrm{Ext}(G,M)$, there is a composition, called the *Baer sum*, and is defined as follows. Given $(E_i,\varphi_i,\iota_i), i=1,2$, construct

$$E_3 = (E_1 \times_G E_1)/\{(\iota_1(m),-\iota_2(m)) \,:\, m\in M\}.$$

Here, $E_1 \times_G E_1$ is the pullback,

$$E_1 \times_G E_1 = \{(x,y)\in E_1\times E_2 \,:\, \varphi_1(x)=\varphi_2(y)\}.$$

There is a well-defined surjective homomorphism

$$\begin{aligned} \varphi_3 : E_3 &\twoheadrightarrow G \\ [(x,y)] &\mapsto \varphi_1(x)=\varphi_2(y). \end{aligned}$$

The kernel is

$$\ker\varphi_3 = \{(\iota_1(m_1),\iota_2(m_2)) \,:\, m_1,m_2\in M\}/\{(\iota_1(m),-\iota_2(m) \,:\, m\in M\},$$

and may be checked to be central. The isomorphism $\iota_3 : M \to \ker\varphi_3$ is

$$\iota_3(m) = [(\iota_1(m),\iota_2(0)].$$

Then one defines the Baer sum to be

$$[(E_1,\varphi_1,\iota_1)] + [(E_2,\varphi_2,\iota_2)] = [(E_3,\varphi_3,\iota_3)].$$

It is not hard to verify that the Baer sum is commutative and associative, that $[(G\times M,\pi_G,\iota_M)]$ is the neutral element, and that

$$-[(E,\varphi,\iota)] = [(E,\varphi,-\iota)].$$

So $\mathrm{Ext}(G,M)$ with the Baer sum becomes an abelian group.

Theorem 14.5. *Let G, M be groups, with M abelian. There is a natural isomorphism*

$$\mathrm{Ext}(G, M) \cong H^2(G, M),$$

where M has the trivial G-action.

Proof. Sketch: Check the following:

- For two extensions with respective 2-cocycles σ_1, σ_2 coming from respective sections s_1, s_2, check that the Baer sum has 2-cocycle $\sigma_1 + \sigma_2$ for the section (s_1, s_2). Deduce that there is a group homomorphism $\Phi : \mathrm{Ext}(G, M) \to H^2(G, M)$.
- Injectivity: If $\Phi([E, \varphi, \iota)]) = 0$ in $H^2(G; M)$, i.e., the 2-cocycle for a section s is a coboundary $d^1 f$, construct a modified section s' which is a homomorphism, thereby showing that (E, φ, ι) is trivial.
- Manually construct $\Psi : H^2(G; M) \to \mathrm{Ext}(G, M)$ such that $\Phi \circ \Psi = \mathrm{id}_{H^2(G;M)}$, so Φ is surjective.

□

We shall use some general machinery of (co)homology of G as a **black box** to deduce the following result.

Theorem 14.6. *If G is a perfect group, and (E, φ) is a universal central extension of G, then $\ker \varphi$ is naturally isomorphic to $H_2(G; \mathbb{Z})$, where $\mathbb{Z}$ has the trivial G-action.*

Proof. (Sketch.) General theory gives a *universal coefficient theorem*, stating that there is a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{j-1}(G, \mathbb{Z}); M) \to H^j(G; M) \to \mathrm{Hom}_{\mathbb{Z}\text{-mod}}(H_j(G, \mathbb{Z}), M) \to 0,$$

relating cohomology of G with coefficients in M to homology of G with trivial coefficients. As G is perfect, $G_{\mathrm{ab}} = 0$, and Example 14.1 shows that $H_1(G; \mathbb{Z}) = 0$. Then the $j = 2$ part of the universal coeffcients theorem gives an isomorphism

$$H^2(G; M) \cong \mathrm{Hom}_{\mathbb{Z}\text{-mod}}(H_2(G, \mathbb{Z}), M), \tag{14.13}$$

for all M (with trivial G-action).

Also, by Theorem 14.1, the universal central extension (E, φ) of G satisfies: (i) E is perfect, and (ii) E does not have any non-trivial central extensions.

- As with G, (i) implies $H_1(E;\mathbb{Z}) = 0$. We also have $H_0(G;\mathbb{Z}) = \mathbb{Z}$ (Example 14.1), then the $j = 1$ part of the universal coefficients theorem gives $H^1(E;M) = 0$. (The $\mathrm{Ext}^1_{\mathbb{Z}}$ term is trivial as all extensions of $\mathbb{Z}$ are trivial — exercise.)
- From (ii) and Theorem 14.5, $H^2(E;M) = 0$ for all M.

Next, for the exact sequence $1 \to \ker\varphi \to E \xrightarrow{\varphi} G \to 1$, there is an *inflation-restriction sequence* sequence (Rosenberg Theorem 4.1.20),

$$\cdots H^1(E;M) \to H^1(\ker\varphi;M) \xrightarrow{\partial} H^2(G;M) \to H^2(E;M) \to \cdots .$$

We had just learned that $H^1(E;M) = 0 = H^2(E;M)$, so the above sequence gives isomorphisms

$$\partial : H^1(\ker\varphi;M) \to H^2(G;M)$$

for all M (with trivial G-action). We also recall from Example 14.2 that

$$H^1(\ker\varphi;M) = \mathrm{Hom}_{\mathbb{Z}}(\ker\varphi;M).$$

Combining the last two isomorphisms with (14.13), we have deduced that

$$\mathrm{Hom}_{\mathbb{Z}\text{-mod}}(\ker\varphi;M) \cong \mathrm{Hom}_{\mathbb{Z}\text{-mod}}(H_2(G,\mathbb{Z}),M)$$

for all M. This forces $\ker\varphi \cong H_2(G;\mathbb{Z})$ as claimed. □

Corollary 14.7. *For a unital ring R, there is a natural isomorphism $K_2(R) \cong H_2(\mathrm{E}(R);\mathbb{Z})$.*

Proof. By Corollary 14.3, $K_2(R)$ is the kernel of a universal central extension of $\mathrm{E}(R)$. Then Theorem 14.6 applies to give the claim. □

In the language of homology of groups, we could use Example 14.2 to write

$$K_1(R) = \mathrm{GL}(R)_{\mathrm{ab}} \cong H_1(\mathrm{GL}(R);\mathbb{Z}).$$

So Corollary 14.7 is a corresponding statement for $K_2(R)$. With a lot of hindsight, we could define

$$K_3(R) = H_3(\mathrm{St}(R);\mathbb{Z}),$$

and show that the K-theory long exact sequence can be extended to start from K_3. But the reason this works requires systematically understanding how to define $K_n(R)$ for all $n \geq 3$ in the first place.