4 (Co)limits

Mimicking limits in analysis (but differing a lot in substance!), there is a general concept of (co)limit in category theory.

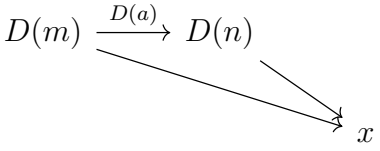
One starts with a functor $D : \mathcal{I} \to \mathcal{C}$, called a *diagram in* $\mathcal{C}$. The category $\mathcal{I}$ is the *indexing* category, whose arrows are mapped by D to become morphisms in $\mathcal{C}$. Intuitively, the colimit of D is some object of $\mathcal{C}$ which the diagram "converges towards".

We will always assume that $\mathcal{I}$ *is small.* A typical example is $\mathcal{I} = \mathbb{N}$ regarded as a category with natural numbers as objects, with an arrow $m \xrightarrow{a} n$ whenever $m \leq n$. In this case, D is a $\mathcal{C}$-valued sequence.

Here are the formal definitions. A *cone from a diagram* $D : \mathcal{I} \to \mathcal{C}$ is collection of morphisms

$$\{D(n) \to x\}_{n \in \mathrm{Ob}(\mathcal{I})}$$

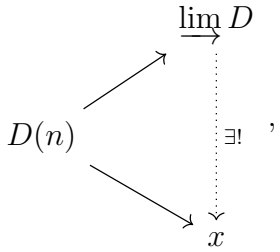
(into some object $x \in \mathrm{Ob}(\mathcal{C})$), such that



$\forall\, m \overset{a}{n}$ in $\mathcal{I}$,

commutes. It is convenient to simply say "the cone x", but it is important to remember the data of the morphisms to x.

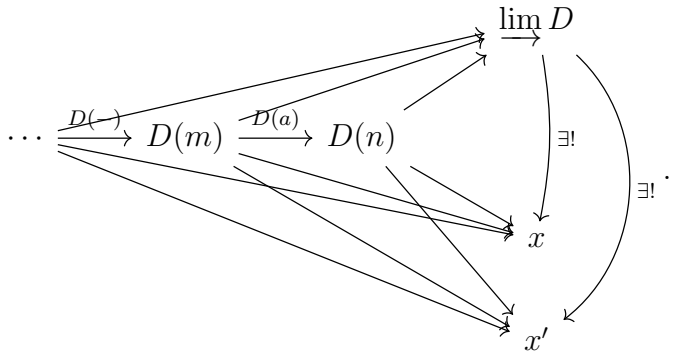
A *colimit* of D is a cone $\varinjlim D$ from D, which is *universal* in the sense that for any other cone x from D, there is a unique morphism $\varinjlim D \to x$ making



, $\quad \forall\, n \in \mathrm{Ob}(\mathcal{I}),$

commute.

So every cone from D factors through $\varinjlim D$; pictorially:



Because of universality, $\varinjlim D$ is unique up to (unique) isomorphism, and the morphisms into $\varinjlim D$ are modified according to that isomorphism.

Example 4.1. Take $\mathcal{C} = \mathbf{Set}$. For a functor $D : \mathcal{I} \to \mathbf{Set}$, its colimit can be chosen to be

$$\varinjlim D = \left(\coprod_{n \in \mathrm{Ob}(\mathcal{I})} D(n) \right) / \sim, \tag{4.1}$$

where $\sim$ is the equivalence relation generated by

$$D(m) \ni x \sim D(a)(x) \in D(n) \text{ whenever } m \xrightarrow{a} n.$$

Equipped with the equivalence class maps $D(n) \to \varinjlim D$, we check that the conditions to be a colimit of D are satisfied. Informally, each arrow in $\mathcal{I}$ is a gluing relation to be imposed in the colimit, and (4.1) achieves this without introducing any extra relations.

Example 4.2. Here is an example which appears in K-theory. Let R be a ring, and $M_n(R)$ be the ring of $n \times n$ matrices with entries in R. Define a functor $M_{(-)}(R) : \mathbb{N} \to \mathbf{Ring}$ as follows. For each arrow $m \xrightarrow{a} n$, set

$$\begin{aligned} M_{(-)}(a) : M_m(R) &\to M_n(R) \\ T_{(m)} &\mapsto \mathrm{diag}(T_{(m)}, 0_{n-m}). \end{aligned}$$

Let $M_\infty(R)$ be the ring of $\mathbb{N} \times \mathbb{N}$ matrices with only finitely-many non-zero entries. It comes with similar upper-left-corner inclusion morphisms

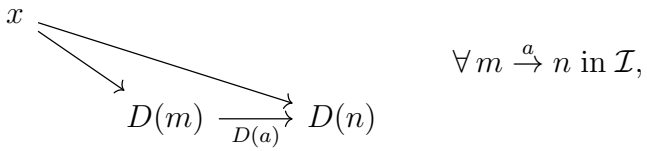
$$M_n(R) \hookrightarrow M_\infty(R). \tag{4.2}$$

Note that $M_\infty(R)$ is the increasing union of the $M_n(R)$. Then we may verify that $M_\infty(R)$, together with the inclusion morphisms (4.2), is a colimit of $M_{(-)}(R)$. (Exercise)

Dually, a *cone to a diagram* $D : \mathcal{I} \to \mathcal{C}$ is a collection of morphisms

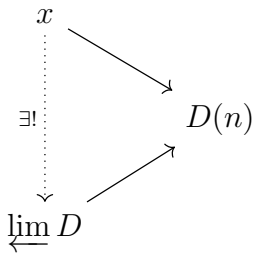
$$\{x \to D(n)\}_{n \in \mathrm{Ob}(\mathcal{I})},$$

(out of some object $x \in \mathrm{Ob}(\mathcal{C})$), such that



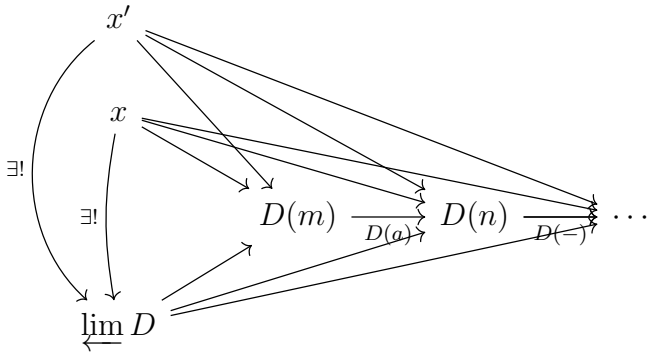
commutes. We often simply say "a cone x", with the morphisms out of x implicit.

A *limit* of D is a cone $\varprojlim D$ to D, which is *universal* in the sense that if x is any other cone to D, then there is a unique morphism $x \to \varprojlim D$ such that



commutes for all $n \in \mathrm{Ob}(\mathcal{I})$.

So every cone to D factors through $\varprojlim D$; pictorially:



Example 4.3. Take $\mathcal{C} = \mathbf{Set}$. For a diagram $D : \mathcal{I} \to \mathbf{Set}$, a limit can be chosen to be

$$\varprojlim D = \left\{ (x_n) \in \prod_{n \in \mathrm{Ob}(\mathcal{I})} D(n) \ : \text{for all arrows } m \xrightarrow{a} n,\ x_n = D(a)(x_m) \right\}.$$

This is a subset of the cartesian product, and the maps $\varprojlim D \to D(n)$ are given by the projections to the n-th component.

Warning: It is not always the case that $D : \mathcal{I} \to \mathcal{C}$ has a colimit or limit, unless $\mathcal{C}$ has some special properties. Even when a (co)limit exists, it may be difficult to describe explicitly.

4.1 Some continuity results

Proposition 4.1. *Let $D : \mathcal{I} \to \mathcal{C}$ be a diagram in a category $\mathcal{C}$.*

1. *Suppose D has a limit*

$$\{p_n : \varprojlim D \to D(n)\}_{n \in \mathrm{Ob}(\mathcal{I})}.$$

Then for each $c \in \mathrm{Ob}(\mathcal{C})$, the diagram $\mathrm{Hom}_{\mathcal{C}}(c, D(-)) : \mathcal{I} \to \mathbf{Set}$ has limit

$$\{(p_n)_* : \mathrm{Hom}_{\mathcal{C}}(c, \varprojlim D) \to \mathrm{Hom}_{\mathcal{C}}(c, D(n))\}_{n \in \mathrm{Ob}(\mathcal{I})}.$$

2. *Suppose D has a colimit*

$$\{p_n : D(n) \to \varinjlim D\}_{n \in \mathrm{Ob}(\mathcal{I})}.$$

Then for each $c \in \mathrm{Ob}(\mathcal{C})$, the diagram[3] $\mathrm{Hom}_{\mathcal{C}}(D(-), c) : \mathcal{I} \to \mathbf{Set}$ *has limit*

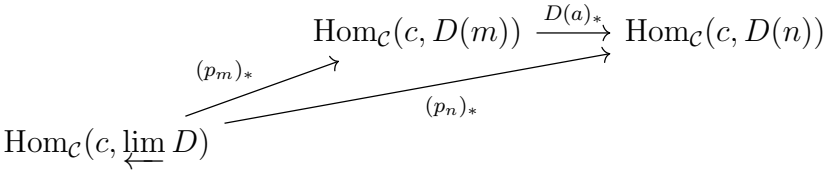
$$\{(p_n)^* : \mathrm{Hom}_{\mathcal{C}}(\varinjlim D, c) \to \mathrm{Hom}_{\mathcal{C}}(D(n), c)\}_{n \in \mathrm{Ob}(\mathcal{I})}.$$

Note that the diagrams $\mathrm{Hom}_{\mathcal{C}}(c, D(-))$ and $\mathrm{Hom}_{\mathcal{C}}(D(-), c)$ have limits (Example 4.3). Prop. 4.1 is often summarized by saying that $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is "continuous".

Proof. (First claim.)

[3] It may be better to say that $\mathrm{Hom}_{\mathcal{C}}(D(-), c)$ is a diagram from $\mathcal{I}^{\mathrm{op}}$ to **Set**.

- By assumption, $\varprojlim D$ exists. We apply the functor $\mathrm{Hom}_{\mathcal{C}}(c, -)$ to the cone property of $\varprojlim D$ to get commuting diagrams



, $\qquad m \xrightarrow{a} n.$

  Thus

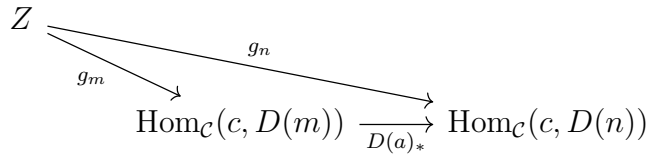
$$\left\{(p_n)_* : \mathrm{Hom}_{\mathcal{C}}(c, \varprojlim D) \to \mathrm{Hom}_{\mathcal{C}}(c, D(n))\right\}_{n \in \mathrm{Ob}(\mathcal{I})}$$

  is a cone to the diagram $\mathrm{Hom}_{\mathcal{C}}(c, D(-)) : \mathcal{I} \to \mathbf{Set}$, and we will show that it is universal, thus a limit of $\mathrm{Hom}_{\mathcal{C}}(c, D(-))$.

- Let

$$\left\{g_n : Z \to \mathrm{Hom}_{\mathcal{C}}(c, D(n))\right\}_{n \in \mathrm{Ob}(\mathcal{I})}$$

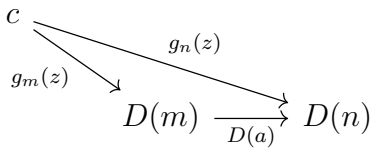
  be another cone to the diagram $\mathrm{Hom}_{\mathcal{C}}(c, D(-))$, i.e.,



  commutes. So for each element $z \in Z$, the morphisms

$$g_n(z) \in \mathrm{Hom}_{\mathcal{C}}(c, D(n)), \qquad D(a) \circ g_m(z) \in \mathrm{Hom}_{\mathcal{C}}(c, D(n))$$

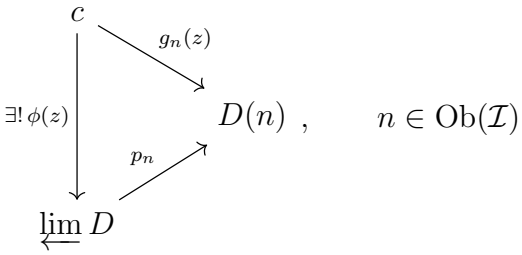
  coincide, i.e.,



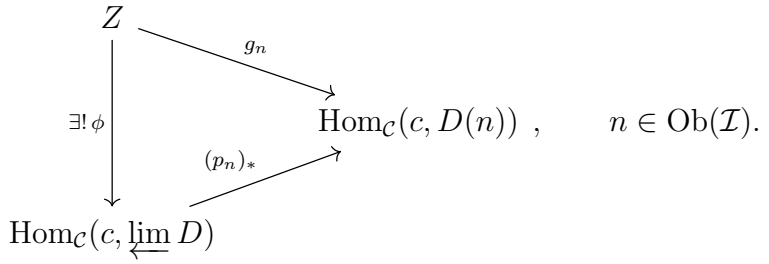
  commutes.

- The last diagram says that c (together with the $g_n(z)$) is a cone to D. By universality of $\varprojlim D$, there is a unique morphism $\phi(z) \in \mathrm{Hom}_{\mathcal{C}}(c, \varprojlim D)$

such that



commute. All these $\phi(z)$ assemble into the unique function $\phi : Z \to \mathrm{Hom}_{\mathcal{C}}(c, \varprojlim D)$ ensuring commutativity of



Therefore, $\mathrm{Hom}_{\mathcal{C}}(c, \varprojlim D)$ is universal as claimed.

(Second claim.) By assumption, D has a colimit

$$\{p_n : D(n) \to \varinjlim D\}_{n \in \mathrm{Ob}(\mathcal{I})}.$$

Since the diagram $\mathcal{I} \xrightarrow{D} \mathcal{C} \xrightarrow{\mathrm{id}} \mathcal{C}^{\mathrm{op}}$ has reversed arrows compared to those of D, we deduce that

$$\{p_n^{\mathrm{op}} : (\varinjlim D)^{\mathrm{op}} \to D(n)^{\mathrm{op}}\}_{n \in \mathrm{Ob}(\mathcal{I})}$$

is a *limit* of $\mathcal{I} \xrightarrow{D} \mathcal{C} \xrightarrow{\mathrm{id}} \mathcal{C}^{\mathrm{op}}$. Then the previous analysis says that for each $c^{\mathrm{op}} \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$,

$$\left\{(p_n^{\mathrm{op}})_* : \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(c^{\mathrm{op}}, (\varinjlim D) \to \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(c^{\mathrm{op}}, D(n)^{\mathrm{op}})\right\}_{n \in \mathrm{Ob}(\mathcal{I})}$$

is a limit for the diagram $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(c^{\mathrm{op}}, D(-)^{\mathrm{op}})$. Passing back from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}$, we deduce that

$$\left\{(p_n)^* : \mathrm{Hom}_{\mathcal{C}}(\varinjlim D, c) \to \mathrm{Hom}_{\mathcal{C}}(D(n), c)\right\}_{n \in \mathrm{Ob}(\mathcal{I})}$$

is a limit for the diagram $\mathrm{Hom}_{\mathcal{C}}(D(-), c)$, for any $c \in \mathrm{Ob}(\mathcal{C})$. □

Proposition 4.2. *Suppose $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ are in adjunction, $L \dashv R$. Then*

- *If a diagram $D : \mathcal{I} \to \mathcal{C}$ be a has a colimit $\varinjlim D$, then $L \circ D$ has a colimit obtained by applying L to (the morphisms accompanying) $\varinjlim D$.*
- *If a diagram $D : \mathcal{I} \to \mathcal{D}$ has a limit $\varprojlim D$, then $R \circ D$ has a limit obtained by applying R to (the morphisms accompanying) $\varprojlim D$.*

Proof. Exercise. Use Prop. 4.1 and Yoneda lemma. □

4.2 (Co)limits in presheaf categories

Via the Yoneda embedding, objects of $\mathcal{C}$ become representables in $\mathbf{Set}_{\mathcal{C}}$. Although $\mathbf{Set}_{\mathcal{C}}$ generally contains many other non-representable objects, assuming $\mathcal{C}$ is small:

1. (Density of representables) Any object of $\mathbf{Set}_{\mathcal{C}}$ arises as a colimit of h_c's.
2. (Objectwise colimits) Colimits in $\mathbf{Set}_{\mathcal{C}}$ always exist, and may be computed "objectwise".

Let us spell out these properties in detail.

Density of representables. Let $F \in \mathbf{Set}_{\mathcal{C}}$ be an arbitrary presheaf. We want to exhibit it as a cone from a diagram $D_F : \mathcal{I}_F \to \mathbf{Set}_{\mathcal{C}}$ which lands on the representables. At the object level, we set

$$\mathrm{Ob}(\mathcal{I}_F) = \left\{ h_c \overset{\alpha}{\Rightarrow} F \,:\, c \in \mathrm{Ob}(\mathcal{C}) \right\} = \coprod_{c \in \mathrm{Ob}(\mathcal{C})} \mathrm{Nat}(h_c, F), \tag{4.3}$$

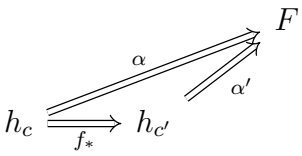
and assign

$$D_F(\alpha) = h_c, \qquad \alpha \equiv h_c \overset{\alpha}{\Rightarrow} H_{c'} \in \mathrm{Ob}(\mathcal{C}).$$

Note that each $D_F(\alpha) = h_c$ tautologically comes with a natural transformation to F, namely, $h_c \overset{\alpha}{\Rightarrow} F$ itself.

The arrows in the indexing category $\mathcal{I}_F$ will be natural transformations $h_c \overset{f_*}{\Rightarrow} h_{c'}$ between representables. (By Yoneda embedding, all such natural transformations are of the form f_* for some unique morphism $c \overset{f}{\to} c'$.) The

information of F dictates which of these arrows appear in $\mathcal{I}_F$, and how they should be organized into hom-sets between the objects α of $\mathcal{I}_F$. Namely,

$h_c \overset{f_*}{\Rightarrow} h_{c'} \in \mathrm{Hom}_{\mathcal{I}_F}\big(h_c \overset{\alpha}{\Rightarrow} F, h_{c'} \overset{\alpha'}{\Rightarrow} F\big)$ iff  commutes. (4.4)

As we can compose commutative diagrams, it follows that $\mathcal{I}_F$ is a category.

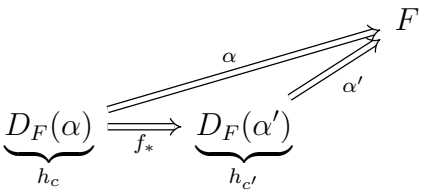
With the objects and arrows of $\mathcal{I}_F$ specified as in (4.3) and (4.4), we set D_F to be the tautological diagram,

$$\begin{aligned} D_F &: \mathcal{I}_F \to \mathbf{Set}_{\mathcal{C}} \\ \big(h_c \overset{\alpha}{\Rightarrow} F\big) &\mapsto h_c \\ \big(h_c \overset{f_*}{\Rightarrow} h_{c'}\big) &\mapsto \big(h_c \overset{f_*}{\Rightarrow} h_{c'}\big). \end{aligned} \tag{4.5}$$

Then

$$\big\{\alpha : \underbrace{D_F(\alpha)}_{h_c} \Rightarrow F\big\}_{\alpha \in \mathrm{Ob}(\mathcal{I}_F)} \tag{4.6}$$

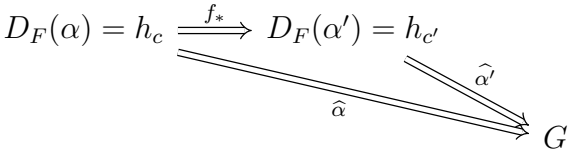
will be a cone from D_F, since by construction,

 commutes $\forall$ arrows $h_c \overset{f_*}{\Rightarrow} h_{c'}$ in $\mathcal{I}_F$. (4.7)

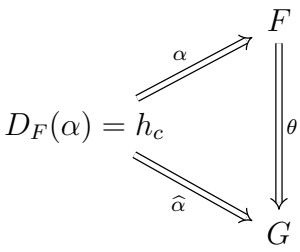
We will show that (4.6) is universal, thus it is a colimit for D_F. So suppose

$$\big\{\widehat{\alpha} : h_c \Rightarrow G\big\}_{\alpha \in \mathrm{Ob}(\mathcal{I}_F)}$$

is another cone from D_F, i.e.,

 , commutes $\forall$ arrows $h_c \overset{f_*}{\Rightarrow} h_{c'}$ in $\mathcal{I}_F$.

We will construct a (unique) $\theta : F \Rightarrow G$ such that



$$\text{commutes } \forall \left(h_c \overset{\alpha}{\Rightarrow} F\right) \in \mathrm{Ob}(\mathcal{I}_F), \tag{4.8}$$

commutes.

Starting from id_c on the left side of (4.8), and recalling the Yoneda bijection formula (2.1), we find that the only possibility is

$$\begin{aligned} &\theta_c : F(c) \to G(c) \\ &\quad I_{c,F}(\alpha) \mapsto I_{c,G}(\widehat{\alpha}), \qquad c \in \mathrm{Ob}(\mathcal{C}). \end{aligned}$$

(Every element of $F(c)$ has the form $I_{c,F}(\alpha)$ for some $\alpha \in \mathrm{Nat}(h_c, F)$, by Yoneda Lemma.)

We verify that $\theta \equiv \{\theta_c\}_{c\in \mathrm{Ob}(\mathcal{C})}$ defines a natural transformation $F \Rightarrow G$. First, for any $H \in \mathrm{Ob}(\mathbf{Set}_{\mathcal{C}})$, the Yoneda Lemma formula (2.2) implies the following commutativity,

$$\begin{array}{ccc} \mathrm{Nat}(h_c, H) & \xleftarrow{(f_*)^*} & \mathrm{Nat}(h_{c'}, H) \\ \downarrow{\scriptstyle I_{c,H}} & & \downarrow{\scriptstyle I_{c',H}} \\ H(c) & \xleftarrow[H(f)]{} & H(c') \end{array} \quad , \qquad \forall c \overset{f_*}{\to} c',$$

because

$$H(f) \circ \beta'(\mathrm{id}_{c'}) \overset{\text{naturality}}{=} \beta' \circ f^*(\mathrm{id}_{c'}) = \beta' \circ f_*(\mathrm{id}_{c'}), \qquad \forall \beta' \in \mathrm{Nat}(h_{c'}, H).$$

This commutativity holds, in particular, for F and G. But F, G are also cones from D_F. So whenever $D_F(\alpha) = h_c \overset{f_*}{\Rightarrow} h_{c'} = D_F(\alpha')$ is an arrow in $\mathcal{I}_F$, meaning that

$$\alpha = \alpha' \circ f_*, \qquad \widehat{\alpha} = \widehat{\alpha'} \circ f_*,$$

we get the commutative diagram

$$\begin{array}{ccc}
\mathrm{Nat}(h_c, F) \ni \alpha & \xleftarrow{(f_*)^*} & \alpha' \in \mathrm{Nat}(h_{c'}, F) \\
\Big\downarrow{\scriptstyle I_{c,F}} & & \Big\downarrow{\scriptstyle I_{c',F}} \\
F(c) \ni I_{c,F}(\alpha) & \xleftarrow[F(f)]{} & I_{c',F}(\alpha') \in F(c') \\
\Big\downarrow{\scriptstyle \theta_c} & & \Big\downarrow{\scriptstyle \theta_{c'}} \\
G(c) \ni I_{c,G}(\widehat{\alpha}) & \xleftarrow[G(f)]{} & I_{c',G}(\widehat{\alpha'}) \in G(c') \\
{\scriptstyle I_{c,G}}\Big\uparrow & & {\scriptstyle I_{c',G}}\Big\uparrow \\
\mathrm{Nat}(h_c, G) \ni \widehat{\alpha} & \xleftarrow[(f_*)^*]{} & \widehat{\alpha'} \in \mathrm{Nat}(h_{c'}, G)
\end{array}$$

Thus

$$\begin{array}{ccc}
F(c) & \xleftarrow{F(f)} & F(c') \\
\Big\downarrow{\scriptstyle \theta_c} & & \Big\downarrow{\scriptstyle \theta_{c'}} \\
G(c) & \xleftarrow[G(f)]{} & G(c')
\end{array}, \qquad \text{commutes } \forall \text{ arrows } h_c \overset{f_*}{\Rightarrow} h_{c'} \text{ in } \mathcal{I}_F. \tag{4.9}$$

Now, for any morphism $c \xrightarrow{f} c'$ in $\mathcal{C}$, either

- $F(c)$ is empty, or
- $h_c \overset{f_*}{\Rightarrow} h_{c'}$ appears as an arrow in $\mathcal{I}_F$ (due to Yoneda Lemma).

So actually, (4.9) commutes for all morphisms $c \xrightarrow{f} c'$ in $\mathcal{C}$. This shows that θ is a natural transformation.

Finally, we check that the natural transformation θ does make (4.8) commute. Indeed, for any arbitrary $g \in h_c(c_1)$, we may use Eq. (2.5) to deduce that

$$\begin{aligned}
\theta_{c_1} \circ \alpha_{c_1}(g) &= \theta_{c_1} \circ F(g)(\alpha_c(\mathrm{id}_c)) \\
&= G(g) \circ \theta_c(I_{c,F}(\alpha)) && \text{(Naturality)} \\
&= G(g) \circ I_{c,G}(\widehat{\alpha}) && \text{(Definition of } \theta) \\
&= \widehat{\alpha}_{c_1}(g).
\end{aligned}$$

To summarize, we have proved the following theorem:

Theorem 4.3. *Let $\mathcal{C}$ be a small category. An arbitrary presheaf $F \in \mathrm{Ob}(\mathbf{Set}_{\mathcal{C}})$ is a colimit of its tautological diagram of representables, D_F (given in* (4.5)*), with the accompanying morphisms given by* (4.6)*.*

Objectwise colimits. For each $c \in \mathrm{Ob}(\mathcal{C})$, there is an evaluation-at-c functor:

$$\begin{aligned} \mathrm{ev}_c : \mathbf{Set}_{\mathcal{C}} &\to \mathcal{C} \\ F &\mapsto F(c) \\ \big(F \overset{\alpha}{\Rightarrow} G\big) &\mapsto \big(F(c) \overset{\alpha_c}{\to} G(c)\big). \end{aligned}$$

Composing a diagram $D : \mathcal{I} \to \mathbf{Set}_{\mathcal{C}}$ with these evaluation functors, we get a collection of diagrams

$$\begin{aligned} &\mathrm{ev}_c \circ D : \mathcal{I} \to \mathbf{Set}, \\ &\qquad n \mapsto D(n)(c) \\ &\big(m \overset{a}{\to} n\big) \mapsto \Big(D(m)(c) \overset{D(a)_c}{\longrightarrow} D(n)(c)\Big), \qquad c \in \mathrm{Ob}(\mathcal{C}), \end{aligned} \tag{4.10}$$

each of which has a colimit (Example 4.1). In other words, for each $c \in \mathrm{Ob}(\mathcal{C})$, we have

$$\big\{\eta_c^{(n)} : \mathrm{ev}_c \circ D(n) \to \varinjlim \mathrm{ev}_c \circ D\big\}_{n \in \mathrm{Ob}(\mathcal{I})},$$

making

$$\begin{array}{ccc} & & \varinjlim \mathrm{ev}_c \circ D \\ & \overset{\eta_c^{(m)}}{\nearrow} \quad \overset{\eta_c^{(n)}}{\nearrow} & \\ \mathrm{ev}_c \circ D(m) & \xrightarrow[\mathrm{ev}_c \circ D(a)]{} & \mathrm{ev}_c \circ D(n) \end{array} \qquad , \qquad \forall\, m \overset{a}{\to} n \tag{4.11}$$

commute, and is universal. The notation $\eta_c^{(n)}$ is meant to suggest that $\{\eta_c^{(n)}\}_{c \in \mathrm{Ob}(\mathcal{C})}$ are natural transformations for each $n \in \mathrm{Ob}(\mathcal{I})$, a fact which will follow from the proof of the next result.

Theorem 4.4. *Let $\mathcal{C}$ be a small category. Then any diagram $D : \mathcal{I} \to \mathbf{Set}_{\mathcal{C}}$ has a colimit. In fact, a colimit is obtained by the "objectwise" formula*

$$\big(\varinjlim D\big)(c) = \varinjlim \mathrm{ev}_c \circ D, \qquad c \in \mathrm{Ob}(\mathcal{C}), \tag{4.12}$$

together with the collection of natural transformations $\eta^{(n)} : D(n) \Rightarrow \varinjlim D$, where $\eta_c^{(n)}$, are the morphisms (4.11) *accompanying the c-th objectwise colimit.*

Proof. The object-level assignment for $\varinjlim D : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is given by (4.12). To make it a functor, we need to specify its morphism-level assignment. This requires the property that D is a functor, which says that:

- There is a collection of contravariant functors

$$D(n) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}, \qquad n \in \mathrm{Ob}(\mathcal{I});$$

- Each arrow $m \xrightarrow{a} n$ in $\mathcal{I}$ is functorially assigned to a natural transformation

$$D(a) : D(m) \Rightarrow D(n).$$

  Each $D(a)$ is itself a collection of functions $\{D(a)_c\}_{c \in \mathrm{Ob}(\mathcal{C})}$, and they make

$$\begin{array}{ccc} D(m)(c) & \xleftarrow{D(m)(f)} & D(m)(c') \\ {\scriptstyle D(a)_c}\downarrow & & \downarrow{\scriptstyle D(a)_{c'}} \\ D(n)(c) & \xleftarrow{D(n)(f)} & D(n)(c') \end{array}, \qquad \forall\, c \xrightarrow{f} c',\ m \xrightarrow{a} n, \tag{4.13}$$

  commute.

The construction of $\varinjlim D$ as a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ proceeds as follows.

- Rewrite the commuting diagrams (4.13) in ev_c notation,

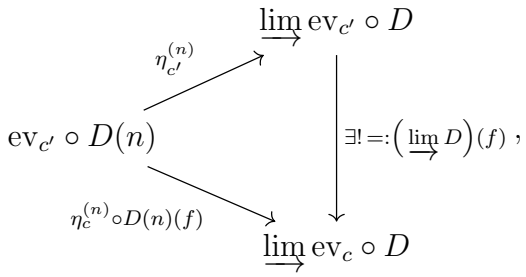
$$\begin{array}{ccc} \mathrm{ev}_c \circ D(m) & \xleftarrow{D(m)(f)} & \mathrm{ev}_{c'} \circ D(m) \\ \downarrow{\scriptstyle \mathrm{ev}_c \circ D(a)} & & \downarrow{\scriptstyle \mathrm{ev}_{c'} \circ D(a)} \\ \mathrm{ev}_c \circ D(n) & \xleftarrow{D(n)(f)} & \mathrm{ev}_{c'} \circ D(n) \\ \downarrow{\scriptstyle \eta_c^{(n)}} & & \downarrow{\scriptstyle \eta_{c'}^{(n)}} \\ \varinjlim \mathrm{ev}_c \circ D & \xleftarrow[\exists?]{} & \varinjlim \mathrm{ev}_{c'} \circ D \end{array} \quad \begin{array}{l} \eta_c^{(m)} \\ \\ \\ \\ \\ \eta_{c'}^{(m)}, \end{array} \qquad c \xrightarrow{f} c',\ m \xrightarrow{a} n. \tag{4.14}$$

  Above, we added the extra vertical morphisms $\eta_-^{(-)}$ into the objectwise colimits.

- For each fixed $c \xrightarrow{f} c'$, the commuting diagram (4.14) says that

$$\left\{\eta_c^{(n)} \circ D(n)(f) : \mathrm{ev}_{c'} \circ D(n) \to \varinjlim \mathrm{ev}_c \circ D\right\}_{n \in \mathrm{Ob}(\mathcal{I})}$$

is a cone from the the diagram $\mathrm{ev}_{c'} \circ D$. By the universal property of the objectwise colimit $\varinjlim \mathrm{ev}_{c'} \circ D$, there is a unique morphism making the following commute



$$n \in \mathrm{Ob}(\mathcal{I}). \tag{4.15}$$

Then, for a composition $c \xrightarrow{f} c' \xrightarrow{g} c''$, uniqueness in (4.15) leads to

$$\big(\varinjlim D\big)(g \circ f) = \big(\varinjlim D\big)(f) \circ \big(\varinjlim D\big)(g), \qquad \big(\varinjlim D\big)(\mathrm{id}_c) = \mathrm{id}_{\varinjlim \mathrm{ev}_c \circ D},$$

showing that $\varinjlim D$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

Because the morphisms $(\varinjlim D)(f)$ of (4.15) make the bottom square of (4.14) commute, i.e.,

$$\begin{array}{ccc} D(n)(c) & \xleftarrow{D(n)(f)} & D(n)(c') \\ {\scriptstyle \eta^{(n)}_{c}}\downarrow & & \downarrow{\scriptstyle \eta^{(n)}_{c'}} \\ \big(\varinjlim D\big)(c) & \xleftarrow[\big(\varinjlim D\big)(f)]{} & \big(\varinjlim D\big)(c') \end{array} \quad , \qquad c \xrightarrow{f} c',\ n \in \mathrm{Ob}(\mathcal{I}), \tag{4.16}$$

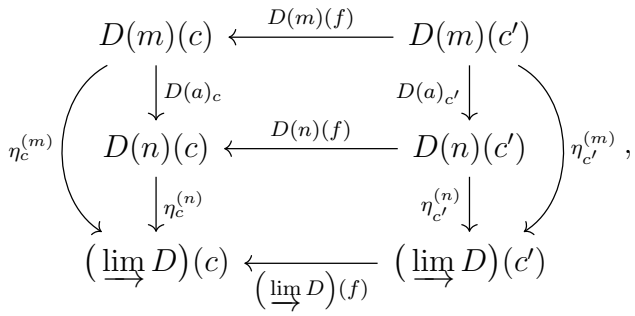
commutes, each $\eta^{(n)} : D(n) \Rightarrow \varinjlim D$ is a natural transformation.

The remaining task is to check that

$$\big\{\eta^{(n)} : D(n) \Rightarrow \varinjlim D\big\}_{n \in \mathrm{Ob}(\mathcal{I})} \tag{4.17}$$

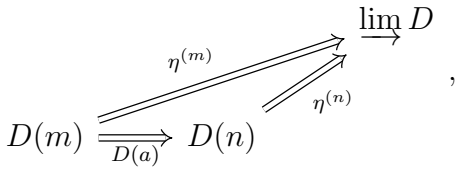
is a colimit for D.

- The combination of (4.16) and (4.13) gives commutativity of



$$\forall\, c \xrightarrow{f} c',\ m \xrightarrow{a} n.$$

Equivalently, the following diagram of natural transformations commutes,



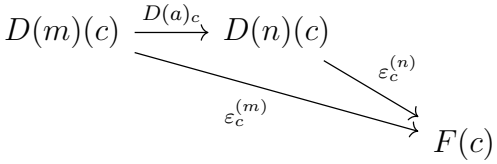
$$\forall\, m \xrightarrow{a} n.$$

Thus (4.17) is a cone from D.

- Let

$$\big\{\varepsilon^{(n)} : D(n) \Rightarrow F\big\}_{n\in \mathrm{Ob}(\mathcal{I})}$$

be another cone from D. In particular, at each $c \in \mathrm{Ob}(\mathcal{C})$,



$$\forall\, m \xrightarrow{a} n \qquad (4.18)$$

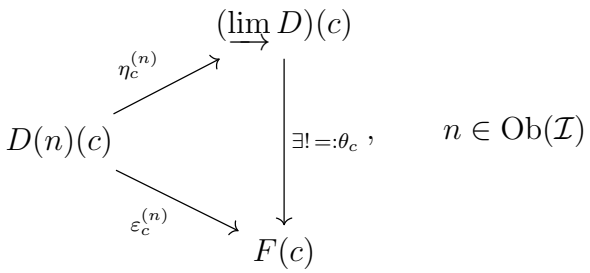
commutes. So

$$\big\{\varepsilon_c^{(n)} : D(n)(c) \to F(c)\big\}_{n\in \mathrm{Ob}(\mathcal{I})}$$

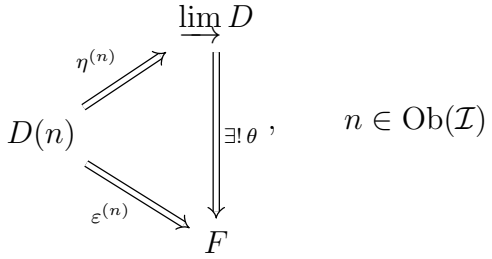
is a cone from the diagram $\mathrm{ev}_c \circ D$.

But $(\varinjlim D)(c) = \varinjlim \mathrm{ev}_c \circ D$ is a colimit for $\mathrm{ev}_c \circ D$, so there is a unique

morphism $\theta_c : (\varinjlim D)(c) \to F(c)$ making

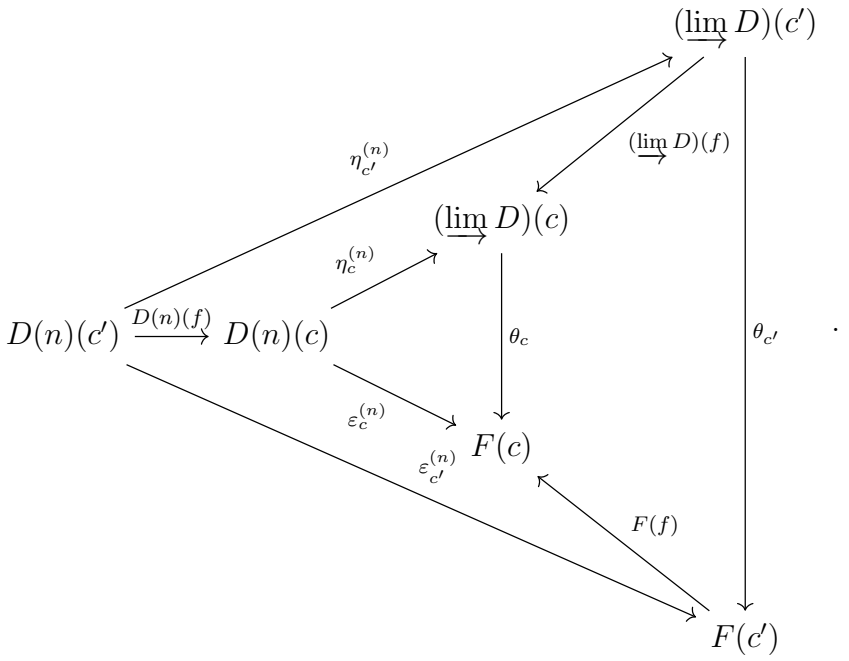


commute. It remains to show that the $\theta_c, c \in \mathrm{Ob}(\mathcal{C})$, assemble into a natural transformation $\varinjlim D \Rightarrow F$. Then we would have



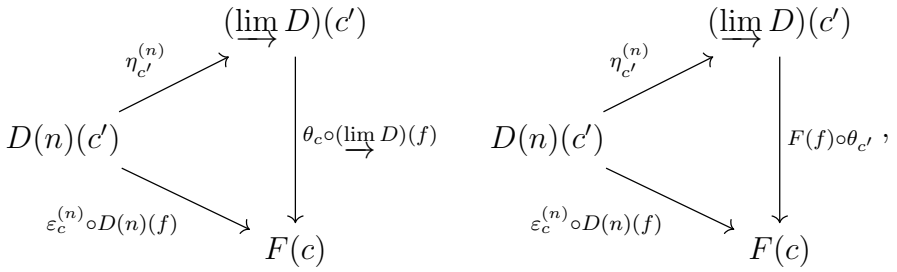
commuting, verifying that (4.17) is a colimit for D.

- For $n \in \mathrm{Ob}(\mathcal{I})$, and $c \xrightarrow{f} c'$, consider the diagram



The trapeziums with $\eta^{(n)}$ and with $\varepsilon^{(n)}$ commute by naturality. The small and big triangles commute by construction of θ_c and $\theta_{c'}$. Naturality of θ is precisely the statement that the trapezium with θ commutes, but we do not know this yet.

- Nevertheless, the above-mentioned commutativities lead us to the following two commuting diagrams



for all $n \in \mathrm{Ob}(\mathcal{I})$, $c \xrightarrow{f} c'$.

- For θ to be a natural transformation, we need to show that

$$\theta_c \circ (\varinjlim D)(f) = F(f) \circ \theta_{c'}, \qquad \forall\, c \xrightarrow{f} c'. \tag{4.19}$$

For each $c \xrightarrow{f} c'$, we can deduce the equality (4.19) by a uniqueness argument:

– The combination of (4.13) and (4.20) says that

$$\begin{array}{ccc} D(m)(c') & \xrightarrow{D(a)_{c'}} & D(n)(c') \\ & {\scriptstyle \varepsilon_c^{(m)} \circ D(m)(f)} \searrow & \swarrow {\scriptstyle \varepsilon_c^{(n)} \circ D(n)(f)} \\ & & F(c) \end{array}, \qquad \forall\, m \xrightarrow{a} n \tag{4.20}$$

commutes. This means that

$$\left\{\varepsilon_c^{(n)} \circ D(n)(f) : D(n)(c') \to F(c)\right\}_{n \in \mathrm{Ob}(\mathcal{I})}$$

is a cone for the diagram $\mathrm{ev}_{c'} \circ D$. Therefore, there is a *unique* map

$$\underbrace{\varinjlim \mathrm{ev}_{c'}}_{(\varinjlim D)(c')} \circ D \to F(c)$$

making

$$\begin{array}{ccc} & & (\varinjlim D)(c') \\ & {\scriptstyle \eta_{c'}^{(n)}} \nearrow & \\ D(n)(c') & & \Big\downarrow {\scriptstyle \exists!} \\ & {\scriptstyle \varepsilon_c^{(n)} \circ D(n)(f)} \searrow & \\ & & F(c) \end{array}, \qquad \forall\, n \in \mathrm{Ob}(\mathcal{I}),$$

commute.

□