# 13 Fundamental Theorem and negative $K$-theory

## 13.1 Split exactness

Here is a quick consequence of Theorem 12.5 and 11.5.

**Proposition 13.1.** *Let $I$ be an ideal in a unital ring $R$. Suppose the exact sequence*

$$0 \longrightarrow I \longrightarrow R \xrightarrow{q} R \underset{s}{\longrightarrow} R/I \longrightarrow 0$$

*admits a splitting homomorphism $s$ (i.e., $q \circ s = \mathrm{id}_{R/I}$). Then there are split exact sequences*

$$0 \longrightarrow K_0(R,I) \xrightarrow{\alpha_*} K_0(R) \underset{s_*}{\overset{q_*}{\rightleftarrows}} K_0(R/I) \longrightarrow 0,$$

*and*

$$0 \longrightarrow K_1(R,I) \xrightarrow{j} K_1(R) \underset{s_*}{\overset{q_*}{\rightleftarrows}} K_1(R/I) \longrightarrow 0$$

*Proof.* For $j = 0, 1, 2$, by functoriality, $q_* \circ s_* = \mathrm{id}_{K_j(R/I)}$. In particular, $q_*$ is surjective. By exactness of the connecting maps $\partial : K_2(R/I) \to K_1(R,I)$ and $\partial : K_1(R/I) \to K_0(R,I)$, they are the zero maps, and then $j : K_1(R,I) \to K_1(R)$ and $(\alpha \circ \tilde{\iota})_* : K_0(I) \to K_0(R)$ are injective. □

*Remark.* (Exercise.) It is possible to directly show the split exactness of the $K_0$ and $K_1$ sequences in Prop. 13.1. Also, the excision property for $K_0$ shows that

$$0 \to K_0(I) \xrightarrow{(\alpha\circ\tilde{\iota})_*} K_0(R) \longleftrightarrow K_0(R/I) \longrightarrow 0$$

is split exact as well.

*Remark.* As the $K$-groups are abelian, the split exact $K_j$ sequences in Prop. 13.1 imply the direct sum decomposition

$$K_j(R) \cong K_j(R,I) \oplus K_j(R/I). \qquad \text{(split case)}$$

### 13.1.1 Polynomial rings and nilpotent $K$-group

For a unital ring $R$, we write $R[z]$ for the ring of polynomials in a formal variable $z$ with coefficients in $R$. Similarly, $R[z^{\pm}]$ denotes the ring of Laurent polynomials in $z, z^{-1}$ with coefficients in $R$. We write $(z)$ for the the ideal generated by the monomial $z$, and $(z-1)$ and $(z^{-1}-1)$ for the ideals generated by the polynomials $z-1$ and $z^{-1}-1$ respectively. There are short exact sequences,

$$\begin{aligned} 0 \to (z) \to & R[z] \stackrel{\mathrm{ev}_0}{\to} R \to 0, \\ 0 \to (z-1) \to & R[z] \stackrel{\mathrm{ev}_1}{\to} R \to 0, \\ 0 \to (z^{-1}-1) \to & R[z^{-1}] \stackrel{\mathrm{ev}_1}{\to} R \to 0, \\ 0 \to (z-1) \to & R[z^{\pm}] \stackrel{\mathrm{ev}_1}{\to} R \to 0, \end{aligned}$$

with splitting homomorphisms taking $r \in R$ to the degree-0 polynomial $r$.

Because of the split exactness of $K_0, K_1$ (Prop. 13.1), there is a canonical copy of $K_j(R)$ in $K_j(R[z])$, in $K_j(R[z^{-1}])$, and in $K_j(R[z^{\pm}])$, and we have splittings

$$\begin{aligned} K_j(R[z]) &= K_j\big(R[z],(z)\big) \oplus K_j(R), \\ K_j(R[z]) &= K_j\big(R[z],(z-1)\big) \oplus K_j(R), \\ K_j(R[z^{-1}]) &= K_j\big(R[z^{-1}],(z^{-1}-1)\big) \oplus K_j(R), \\ K_j(R[z^{\pm}]) &= K_j\big(R[z^{\pm}],(z-1)\big) \oplus K_j(R), \qquad j = 0, 1. \end{aligned} \tag{13.1}$$

The *Fundamental Theorem for* $K_1$ is a statement about $K_1(R[z^{\pm}])$, particularly, a certain further decomposition of the summand $K_1\big(R[z^{\pm}],(z-1)\big)$; see Theorem 13.9. To state and understand it, we need some preparatory notation, as well as a generalization of $K_0$ described in Section 13.2.

- We write

$$\begin{aligned} NK_1(R) &:= \ker\big(K_1(R[z]) \stackrel{(\mathrm{ev}_0)_*}{\longrightarrow} K_1(R)\big) \cong K_1(R[z],(z)), \\ NK_{1,+}(R) &:= \ker\big(K_1(R[z]) \stackrel{(\mathrm{ev}_1)_*}{\longrightarrow} K_1(R)\big) \cong K_1(R[z],(z-1)), \\ NK_{1,-}(R) &:= \ker\big(K_1(R[z^{-1}]) \stackrel{(\mathrm{ev}_1)_*}{\longrightarrow} K_1(R)\big) \cong K_1(R[z],(z^{-1}-1)). \end{aligned} \tag{13.2}$$

  The $N$ stands for "nilpotent", because of the connection with nilpotent endomorphisms as discussed later.

- Suprisingly, the extra summand $K_1(R[z^{\pm}], (z-1))$ in $K_1(R[z^{\pm}])$ contains a copy of $K_0(R)$ (see Definition 19).

**Definition 19.** The *Bott map* is the homomorphism

$$\begin{aligned} \beta : K_0(R) &\to K_1(R[z^{\pm}]) \\ [p] &\mapsto [pz + (1_n - p)] = [1_n + p(z-1)]. \end{aligned} \tag{13.3}$$

Some details:

- If $p, \tilde{p}$ are conjugate idempotents via $u \in \mathrm{GL}_n(R)$, then regarding $u$ as a degree-0 polynomial in $R[z^{\pm}]$,

$$1_n + \tilde{p}(z-1) = u(1_n + p(z-1))u^{-1}$$

  represents the same $K_1(R[z^{\pm}])$ class as $1_n + p(z-1)$.

- Formula (13.3) is additive on $\mathrm{Idem}(R)$, and induces an abelian group homomorphism $K_0(R) \to K_1(R[z^{\pm}])$. Noting that the inverse of $[1_n + p(z-1)]$ is $[1_n p(z^{-1} - 1)]$, the formula for the Bott map is

$$[p_1] - [p_2] \mapsto [(1_n + p_1(z-1))(1_n + p_2(z^{-1} - 1)].$$

*Remark* 10. If we consider the variable $z$ as taking values in the circle $S^1$ of unit complex numbers, then the polynomials in $z, z^{-1}$ are sup-norm dense in the ring of continuous functions $C(S^1)$ (Stone–Weierstrass). If we were dealing with topological $K$-theory, then introducing an extra $C(S^1)$ factor is "topological suspension". The "reduced" version ignores the part coming from evaluation at a basepoint. The reduced suspensions $K^{\mathrm{top}}_{n+1}(-) = K^{\mathrm{top}}_n(\text{reduced suspension}(-))$. The key theorem of topological $K$-theory is that the Bott map is an *isomorphism*; so $K_0(-) \cong K_1(\text{reduced suspension}(-)) \equiv K_2(-)$. Via the Bott periodicity isomorphisms, one

In the algebraic setting, we may think of $R[z^{\pm}]$ as an algebraic analogue of suspension, and in this case, the Bott map is no longer an isomorphism (mod $K_1(R)$).

## 13.2 Generalization of $K_0$

### 13.2.1 Finitely-generated projective modules

Let $R$ be a unital ring. We say that an $R$-module $V$

- Has *generating set* $T \subseteq V$, if every $v \in V$ is a finite $R$-linear combination of the elements in $T$,
$$v = \sum_{i=1}^{n} t_i r_i, \qquad t_i \in T,\, r_i \in R.$$
- Is *finitely-generated*, if it admits a *finite* generating set.
- Has a *basis* $T \subseteq V$, if $T$ is a generating set whose elements are linearly independent,
$$\sum_{i=1}^{n} t_i r_i = 0 \;\Rightarrow\; r_1, \ldots, r_n = 0.$$
- Is *free*, if it has a basis.
- Is *projective*, if it is a direct summand[23] of a free module.

*Exercise* 13.1. Show that an $R$-module $P$ is projective iff whenever $\eta : V \twoheadrightarrow P$ is a surjective morphism, there exists a morphism $\sigma : P \to V$ such that $\eta \circ \sigma = \mathrm{id}_P$.

Finitely-generated projective $R$-modules can be described in terms of idempotent matrices over $R$, as follows. Let $R^n := R^{\oplus n}$ be a finitely-generated free $R$-module. Via its standard basis, the ring of endomorphisms of $R^n$ is identified with $M_n(R)$. Let $p = p^2 \in \mathrm{M}_n(R)$ be an idempotent. Check that
$$pR^n := \{pv \,:\, v \in R^n\}$$
is a (finite-generated) projective $R$-module.

Conversely, let $P$ be a finitely-generated projective $R$-module, and let $\{t_1, \ldots, t_n\}$ be a generating set for $P$. The map
$$\eta : R^n \to P, \qquad (r_1, \ldots, r_n) \mapsto \sum_{i=1}^{n} t_i r_i$$
is a surjective morphism. By Exercise 13.1, there is an injective morphism $\sigma : P \to R^n$ such that $\eta \circ \sigma = \mathrm{id}_P$. We check that $R^n = \ker \eta \oplus \sigma(P)$. Set $p \in \mathrm{End}(R^n)$ to be
$$p(v) = \sigma \circ \eta(v), \qquad v \in R^n,$$
which satisfies $p^2 = p$. Check that $pR^n = \sigma(P) \cong P$.

[23] Let $V_1, V_2$ be submodules of $V$. We say that $V$ is the direct sum $V_1 \oplus V_2$ if (i) $V_1 + V_2 = V$, and (ii) $V_1 \cap V_2 = \{0\}$.

**Proposition 13.2.** *Let $R$ be a unital ring and $p \in M_n(R)$, $p' \in M_{n'}(R)$ be idempotents. The $R$-modules $pR^n$ and $p'R^{n'}$ are isomorphic iff there exists an invertible matrix $u \in \mathrm{GL}_N(R)$ such that*

$$u(p \oplus 0_{N-n})u^{-1} = p' \oplus 0_{N-n'}. \tag{13.4}$$

*Proof.* $\Leftarrow$ is straightforward. For $\Rightarrow$, by adding some zeroes, we may regard $p, p'$ as idempotents of the same size $n$. The required $u \in \mathrm{GL}_{2n}(R)$ is that in Eq. (12.1) of Prop. 12.1. □

Let fpMod-$R$ denote the set of *isomorphism classes* of finitely-generated projective $R$-modules. Note that each isomorphism class has a representative of the form $pR^n$ with idempotent $p$, so fpMod-$R$ is genuinely a set. Furthermore, it is a monoid under direct sum (of representative modules). Because of Prop. 13.2, there is a canonical monoid isomorphism

$$\text{fpMod-}R \cong \mathrm{Idem}(R).$$

This means that we can equivalently define $K_0(R)$ as the Grothendieck group of the abelian monoid fpMod-$R$.

### 13.2.2 $K_0$ of exact categories

Let **fpMod-**$R$ be the category of finitely-generated projective $R$-modules. Instead of "decategorifying" to fpMod$-(R)$ before defining $K_0(R)$, we could also define $K_0(R)$ in terms of categorical properties of **fpMod-**$R$.

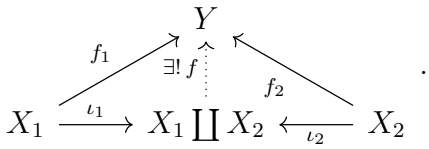
Some terminology before explaining how this works:

- An *additive category* $\mathcal{C}$ has
  - Hom-sets being abelian groups, and composition being bilinear.
  - A distinguished object $0_{\mathcal{C}}$ such that

    $$\mathrm{Hom}_{\mathcal{C}}(X, 0_{\mathcal{C}}) = \{0\} = \mathrm{Hom}_{\mathcal{C}}(0_{\mathcal{C}}, X), \qquad \forall \text{ objects } X.$$

    (We will be sloppy and just write 0 for the object $0_{\mathcal{C}}$, and also for the "zero morphisms" $X \to 0_{\mathcal{C}}$ and $0_{\mathcal{C}} \to X$.)

– A binary operation $\oplus$ on objects, which is both a product and a coproduct. Recall that a coproduct of objects $X_i, i = 1, 2$, is a pair of morphisms $X_i \xrightarrow{\iota_i} X_1 \coprod X_2$, such that given morphisms $X_i \xrightarrow{f_i} Y$, there exists a unique morphism $X_1 \coprod X_2 \xrightarrow{f} Y$ make the following diagram commute,

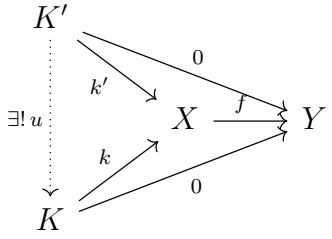



A product is defined dually.

A typical example of an additive category is the category **Mod-**$R$ of $R$-modules, with $\oplus$ the direct sum. The subcategory **pMod-**$R$ of projective $R$-modules is also additive.

- An *abelian* category $\mathcal{A}$ is an additive category such that
  - Every morphism $f$ has a kernel and a cokernel.

    Recall that a kernel of $X \xrightarrow{f} Y$ is a morphism $K \xrightarrow{k} X$ satisfying $f \circ k = 0$, which is universal in the sense that for any other morphism $K' \xrightarrow{k'} X$ satisfying $f \circ k' = 0$, there is a unique $K' \xrightarrow{u} K$ making the following diagram commute:




    A cokernel $Y \xrightarrow{q} Q$ is defined dually.
  - Every monomorphism is the kernel of some morphism, and every epimorphism is the cokernel of some morphism.

    Recall that a $K \xrightarrow{k} X$ is a monomorphism if for any $g, g' \in \mathrm{Hom}_{\mathcal{C}}(Z, K)$,

$$k \circ g = k \circ g' \Rightarrow g = g'.$$

Dually, $Y \xrightarrow{q} Q$ is an epimorphism if for any $g, g' \in \mathrm{Hom}_{\mathcal{C}}(Q, Y)$,

$$g \circ q = g' \circ q \Rightarrow g = g'.$$

These are abstractions of injective and surjective functions, respectively.

Typical examples of abelian categories are the category of abelian groups and the module categories **Mod-**$R$. Here, "kernel" and "cokernel" often also refer to the objects $K$ and $Q$, with the morphisms $k$ and $q$ implicit.

However, **fpMod-**$R$ is generally not an abelian category. For example, consider the morphism

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$$

of (projective) $\mathbb{Z}$-modules. In **Mod-**$R$, this has cokernel being $\mathbb{Z} \xrightarrow{\mathrm{mod}\ 2} \mathbb{Z}_2$, but $\mathbb{Z}_2$ is not a projective $\mathbb{Z}$-module.

- An *exact category*[24] is a full[25] additive subcategory $\mathcal{P}$ of an abelian category $\mathcal{A}$, such that
  - If
    $$0 \to P_1 \to V \to P_2 \to 0 \tag{13.5}$$
    is a short exact sequence[26] in $\mathcal{A}$ with $P_1, P_2 \in \mathrm{Ob}(\mathcal{P})$, then $V \in \mathrm{Ob}(\mathcal{P})$ as well.
  - $\mathcal{P}$ has a small skeleton, i.e., a small full subcategory $\mathcal{P}_0$ such that $\mathcal{P}_0 \hookrightarrow \mathcal{P}$ is an equivalence.

  The *short exact sequences in* $\mathcal{P}$ are the short exact sequences in $\mathcal{A}$ which involve only objects of $\mathcal{P}$.

  A typical example of an exact category is **fpMod-**$R$ (inside **Mod-**$R$. Any object of **fpMod-**$R$ is isomorphic to some element in the set of modules $pR^n$ with idempotent $p$. By Exercise 13.1, a short exact sequence (13.5) with $P_1, P_2 \in \mathrm{Ob}(\mathcal{P})$ splits, then $V \cong P_1 \oplus P_2$ is another object in $\mathcal{P}$.

[24] The meaning of "exact category" may differ from author to author.

[25] Meaning the hom-sets are the same as those in the ambient category.

[26] For concrete module categories, this means image of a morphism equals the kernel of the next morphism. For abstract abelian categories, "image" would need a bit more explanation.

**Definition 20.** Let $\mathcal{P}$ be an exact category. Then $K_0(\mathcal{P})$ is the free abelian group with generators $[P]$, where $[P] \in \mathrm{Ob}(\mathcal{P}_0)$, subject to

- $[P] = [P_1] + [P_2]$ if
$$0 \to P_1 \to P \to P_2 \to 0 \tag{13.6}$$
is a short exact sequence in $\mathcal{P}$.

*Remark.* The notation $[P]$ makes sense for any $P \in \mathrm{Ob}(\mathcal{P})$, referring to a generator labelled by an(y) object of $\mathcal{P}_0$ isomorphic to $P$. The special case of (13.6) with $P_1 = 0$ says that $[P] = [P']$ whenever $P, P'$ are isomorphic objects in $\mathcal{P}$, so there is no ambiguity. The smallness condition on $\mathcal{P}_0$ allows for the set-theoretic construction of the group $K_0(\mathcal{P})$.

A functor $F : \mathcal{P} \to \mathcal{P}'$ between exact categories is *exact* if it maps short exact sequences to short exact sequences.

**Proposition 13.3.** *If $F : \mathcal{P} \to \mathcal{P}'$ is an exact functor, then there is an induced homomorphism*
$$F_* : K_0(\mathcal{P}) \to K_0(\mathcal{P}').$$
*This is functorial in the sense that $(\mathrm{Id}_{\mathcal{P}})_* = \mathrm{id}_{K_0(\mathcal{P})}$, and if $G : \mathcal{P}' \to \mathcal{P}''$ is another exact functor,*
$$(G \circ F)_* = G_* \circ F_*.$$

*Proof.* $F_*$ is the map sending each generator $[P]$ to $[F(P)]$. As $F$ is an exact functor, the defining relations for $K_0(\mathcal{P}')$ are respected, and $F_*$ descends to a group homomorphism. The functoriality is straighforward to check. □

**Proposition 13.4.** *Let $R$ be a unital ring. Then $K_0(\mathbf{fpMod}\text{-}R)$ is canonically isomorphic to $K_0(R)$.*

*Proof.* By definition, both $K_0(\mathbf{fpMod}\text{-}R)$ and $K_0(R)$ have fpMod-$R$ as a generating set, although the relations might, a priori, be different. In $K_0(\mathbf{fpMod}\text{-}R)$, the sum $[P_1]+[P_2]$ equals $[P]$ where $P$ is any finitely-generated projective module appearing in a short exact sequence (13.6). For example, we can choose $P = P_1 \oplus P_2$, i.e.,
$$[P_1] + [P_2] = [P_1 \oplus P_2].$$
This is the same as the addition operation in $K_0(R)$.

The Grothendieck group $K_0(R)$ can be described as the free abelian group on fpMod-$R$ subject to $[P_1] = [P_2]$ if $P_1 \cong P_2$, and $[P_1] + [P_2] = [P_1 \oplus P_2]$. These relations are satisfied in $K_0(\mathbf{fpMod}\text{-}R)$ as well.

Conversely, whenever $[P_1], [P_2]$ are such that there is a short exact sequence (13.6), we have, by definition $[P_1] + [P_2] = [P]$ in fpMod-$R$. But such a short exact sequence splits (Exercise 13.1), so $P \cong P_1 \oplus P_2$, thus $[P_1]+[P_2] = [P_1 \oplus P_2]$ holds. These relations are also satisfied in $K_0(R)$. □

To get something genuinely new from Definition 20, we need to apply it to an exact category which is not of the form **fpMod-**$R$.

**Definition 21.** For a unital ring $R$, the *category* $\mathbf{Nil}(R)$ *of nilpotent endomorphisms* has objects being pairs $(P, \nu)$, where $A$ is a nilpotent endomorphism of a finitely-generated projective $R$-module $P$. The hom-sets are

$$\mathrm{Hom}_{\mathbf{Nil}(R)}\big((P,\nu),(P',\nu')\big) = \big\{T \in \mathrm{Hom}_{\mathbf{Mod}\text{-}R}(P,P') \,:\, \nu' T = T\nu\big\}.$$

*Remark.* Dropping the "finitely-generated projective" and "nilpotent" requirements, we recover an abelian category, in which $\mathbf{Nil}(R)$ is an exact subcategory.

Restricting $\mathbf{Nil}(R)$ to those objects having $\nu = 0$, we get **fpMod-**$R$. The inclusion functor **fpMod-**$R \to \mathbf{Nil}(R)$ has left inverse being the "forget-$\nu$" functor $\mathbf{Nil}(R) \to$ **fpMod-**$R$. These are exact functors. Taking $K_0$-groups, functoriality (Prop. 13.3) means that there is a split exact sequence of abelian groups

$$0 \to \underbrace{K_0(\mathbf{fpMod}\text{-}R)}_{=K_0(R)} \to K_0(\mathbf{Nil}(R)) \to \mathrm{Nil}_0(R) \to 0,$$

with $\mathrm{Nil}_0(R)$ the cokernel (quotient group). Via the splitting, we have

$$K_0(\mathbf{Nil}(R)) \cong K_0(R) \oplus \mathrm{Nil}_0(R). \tag{13.7}$$

Explicitly, because of the complemented property of finitely-generated projective $R$-modules, $\mathrm{Nil}_0(R)$ is identified as the subgroup of $K_0(\mathbf{Nil}(R))$ generated by

$$[(R^n, \nu)] - [(R^n, 0)].$$

## 13.3 Bass–Heller–Swan

**Lemma 13.5** (Higman's linearization trick)**.** *Any class in $NK_1(R) \subseteq K_1(R[t])$ has a linear representative of the form $1+\nu z$, where $\nu$ is some nilpotent matrix over $R$. Conversely, any such linear polynomial represents a class in $NK_1(R)$.*

*Proof.* A priori, a class in $NK_1(R) = K_1(R[z],(z))$ is represented by an invertible polynomial of the form

$$1 + \sum_{j=1}^{d} r_j z^j$$

with degree $d > 1$. Here, the coefficients $r_j$ come from some $M_n(R)$. We have the reductions

$$\begin{aligned}
1 + \sum_{j=1}^{d} r_j z^j &= \begin{pmatrix} 1 + \sum_{j=1}^{d} r_j z^j & 0 \\ 0 & 1 \end{pmatrix} && \text{(double matrix size)} \\
&\rightsquigarrow \begin{pmatrix} 1 + \sum_{j=1}^{d} r_j z^j & r_d z^{d-1} \\ 0 & 1_n \end{pmatrix} && \text{(add } r_d z^{d-1} \times \text{(second row) to first row)} \\
&\rightsquigarrow \begin{pmatrix} 1 + \sum_{j=1}^{d-1} r_j z^j & r_d z^{d-1} \\ -z & 1 \end{pmatrix}. && \text{(add } -z \times \text{(second column) to first column)}
\end{aligned}$$

These reductions are implemented by elementary matrices in $\mathrm{E}(R[z])$, so the $K_1(R[z])$ class (thus $NK_1(R)$ class) is unchanged. Iterating this doubling and row/column operation, we get a representative of the form $1 + \nu z$, where $\nu$ is some matrix over $R$. It remains to verify that $\nu$ is a nilpotent.

As $1+\nu z$ is invertible in $\mathrm{GL}(R[z])$, let $\sum_{j=0}^{m} a_m z^m$ be its (two-sided) inverse. Comparing coefficients of $z^j$ in the expansion of

$$(1 + \nu z)(\sum_{j=0}^{m} a_j z^j) = 1 = (\sum_{j=0}^{m} a_j z^j)(1 + \nu z),$$

we have the simultaneous equations

$$\begin{aligned}
a_0 &= 1 \\
\nu a_0 + a_1 &= 0 = a_0 \nu + a_1 \\
&\vdots \\
\nu a_{m-1} + a_m &= 0 = a_{m-1} \nu + a_m \\
\nu a_m &= 0 = a_m \nu.
\end{aligned}$$

The first $m + 1$ equations are solved by $a_j = (-\nu)^j, j = 0 \ldots, m$. Then the final equation becomes the nilpotence property $\nu^{m+1} = 0$.

Conversely, if $\nu$ is a nilpotent matrix over $R$, then it is straightforward to check that $1+\nu z$ has inverse $1+\sum_{j\geq 1}(-\nu)^j z^j$ (which is a finite sum). So $[1+\nu z]$ represents a $K_1(R[z])$ class. But

$$(\mathrm{ev}_0)_*[1+\nu z] = [1] \in K_1(R),$$

so $[1+\nu z] \in \ker(\mathrm{ev}_0)_* = NK_1(R)$. □

**Proposition 13.6.** *$NK_1(R)$ and $NK_{1,\pm}$ are canonically isomorphic to $\mathrm{Nil}_0(R)$.*

*Proof.* As in Lemma 13.5, any class in $NK_1(R)$ has the form $[1_n+\nu z] \in NK_1(R)$, with $\nu$ a nilpotent matrix over $R$. Define

$$\begin{aligned} \delta : NK_1(R) &\mapsto \mathrm{Nil}_0(R) \\ [1+\nu z] &\mapsto [(R^n,\nu)] - [(R^n,0)]. \end{aligned} \tag{13.8}$$

The independence of formula (13.8) from the choice of nilpotent $\nu$ needs some checks[27].

- If $\nu$ is regarded as having size $n+k$, i.e., $1_n+\nu z$ is replaced by $1_{n+k}+\mathrm{diag}(\nu,0_k)z$, then under the map (13.8),

$$\begin{aligned} \delta[1_{n+k}+\mathrm{diag}(\nu,0_k)z] &= [(R^{n+k},\mathrm{diag}(\nu,0_k))] - [(R^{n+k},0_{n+k})] \\ &= [(R^n,\nu)] + [(R^k,0_k)] - [(R^n,0_n)] - [(R^k,0_k)] \\ &= [(R^n,\nu)] - [(R^n,0_n)] = \delta[1_n+\nu z]. \end{aligned}$$

- Suppose $1_n+\mu z$ is elementary, with $\mu z \in M_n((z))$. So $\mu$ is a (nilpotent) $n\times n$ matrix which is zero everywhere except for a single off-diagonal $ij$ entry, $i\neq j$. We can construct a short exact exact sequence

$$0 \to R^{n-1} \xrightarrow{T_1} R^n \xrightarrow{T_2} R \to 0,$$

[27] Although this detail is sometimes omitted, it does not seem easy to establish. An abstract approach can be found in Weibel's book, Prop. III.3.5.3, Example III.3.1.4, where it is shown that $\delta$ is a composition of homomorphisms

$$NK_1(R) = K_1(R[z],(z)) \hookrightarrow K_1(R[z]) \xrightarrow{z\mapsto z^{-1}} K_1(R[z^{\pm}]) \to \mathrm{Nil}_0(R) \subseteq K_0(\mathbf{Nil}(R)).$$

where

$$T_1 : e_k \mapsto \begin{cases} e_k, & k < j, \\ e_{k+1}, & k \geq j, \end{cases} \qquad k = 1, \ldots, n-1,$$

and $T_2 = e_j^*$. We check that there is a short exact sequence

$$0 \to (R^{n-1}, 0) \xrightarrow{T_1} (R^n, \mu) \xrightarrow{T_2} (R, 0) \to 0$$

in $\mathbf{Nil}(R)$, so $[(R^n, \mu)] = [(R^{n-1}, 0)] + [(R, 0)] = [(R^n, 0)]$. Thus

$$\delta[1_n - \mu z] = [(R^n, \mu)] - [(R^n, 0)] = 0. \tag{13.9}$$

Now if $[1_n + \nu z] = [1_n + \nu' z] \in NK_1(R) = K_1(R[z], (z))$ represent the same class but with possibly different nilpotent matrices $\nu, \nu'$, then after stabilization (increasing $n$) and Higman linearization, $(1_n + \nu' z)^{-1}(1_n + \nu' z)$ is a product of $\mathrm{E}(R[z])$-conjugates of elementary matrices of the above $1_n + \mu z$ form. By Higman linearization, the conjugating matrices can themselves be written as a product of elementary matrices of the $1_n + \mu z$ form (increasing $n$ again). According to the prescription (13.8) and the trivialization (13.9), all these $1_n + \mu z$ do not contribute to $\delta$. Thus

$$\delta[1_n + \nu z] = \delta[1_n + \nu' z]$$

without ambiguity.

- Additivity of (13.8) is now straightforward.

The inverse map to (13.8) has the formula

$$\begin{aligned} \tau : \mathrm{Nil}_0(R) &\to NK_1(R) \\ [(R^n, \nu)] - [(R^n, 0)] &\mapsto [1 + \nu z], \end{aligned}$$

where we note that $[1 + \nu z]$ does lie in $NK_1(R)$ by Lemma 13.5. We need to check that $\tau$ is a group homomorphism. So suppose there is a short exact sequence in $\mathrm{Nil}_0(R)$. This means that there is a short exact sequence of finitely-generated free modules

$$0 \to V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3 \to 0,$$

together with nilpotent $\nu_j \in \mathrm{End}(V_j)$, satisfying

$$\nu_2 T_1 = T_1 \nu_1, \qquad \nu_3 T_2 = T_2 \nu_2.$$

If we applied $\tau$ to $x_1 = [(V_1, \nu_1)] - [(V_1, 0)]$ and $x_3 = [(V_3, \nu_3)] - [(V_3, 0)]$ separately, then added the result, we would get $[(1+\nu_1 z) \oplus (1+\nu_3 z)]$. Whereas if we applied $\tau$ directly to $x_2 = [(V2, \nu_2)] - [(V_2, 0)]$ (which equals $x_1 + x_3$ according to the defining relations in $\mathbf{Nil}(R)$), we would get $[1 + \nu_2 z]$. We need to show that $[1 + nu_2 z] = [(1 + \nu_1 z) \oplus (1 + \nu_3 z)]$.

With respect to a splitting $V_2 \cong V_1 \oplus V_3$, we can represent

$$1 + \nu_2 z = \begin{pmatrix} 1 + \nu_1 z & rz \\ 0 & 1 + \nu_3 z \end{pmatrix} = \begin{pmatrix} 1 + \nu_1 z & 0 \\ 0 & 1 + \nu_3 z \end{pmatrix} \begin{pmatrix} 1 & (1 + \nu_1 z)^{-1} rz \\ 0 & 1 \end{pmatrix},$$

with $r$ being some block matrix over $R$. Because

$$(1 + \nu_1 z)^{-1} rz = (1 - \nu_1 z + \nu_1^2 z^2 - \ldots) rz,$$

its entries lie in the ideal $(z)$, so the matrix $\begin{pmatrix} 1 & (1 + \nu_1 z)^{-1} rz \\ 0 & 1 \end{pmatrix}$ lies in $\mathrm{E}(R[z], (z))$, which gets modded out when taking $K_1$ classes. Thus, we have

$$[1 + \nu_2 z] = [1 + \nu_1 z] + [1 + \nu_3 z]$$

in $NK_1(R)$.

For $NK_{1,+}(R)$, we replace the role of $z$ with $z-1$. The Higman linearization Lemma 13.5 proceeds in the same way, and we modify $\delta$ to

$$\begin{aligned} \delta_+ : NK_{1,+}(R) &\to \mathrm{Nil}(R) \\ [1 + \nu(z-1)] &\mapsto [(R^n, \nu)] - [(R^n, 0)]. \end{aligned}$$

Similarly, for $NK_{1,-}(R)$, replace $z$ with $z^{-1} - 1$. □

**Lemma 13.7.** *Any class in $K_1(R[z^{\pm}], (z-1)) \subseteq K_1(R[z^{\pm}])$ can be written as*

$$[z^{-\ell}] + [1_n + A(z-1)],$$

*for some $\ell \in \mathbb{N}$, and $A \in M_n(R)$ with $A(1-A)$ nilpotent.*

*Proof.* Classes in $K_1(R[z^{\pm}], (z-1))$ are represented by $\mathrm{GL}(R[z^{\pm}], (z-1))$ elements, i.e.,

$$1_k + \sum_{j>0} r_j (z-1)^j + \sum_{j>0} r'_j (z^{-1}-1)^j.$$

(coefficients in some $M_k(R)$). After factoring out a power of $1_k \cdot z$, we may assume that it has the form $1 + \sum_{j=1}^{d} r_j (z-1)^j$. By the same kind of reduction argument in Lemma 13.5, after stabilizing and multiplying by elementary matrices in $\mathrm{E}(R[z^{\pm}], (z-1))$, we may reduce to a linear polynomial

$$1_n + A(z-1).$$

Below we write $1_k = 1$ for convenience. We will show that $A(1-A)$ is nilpotent. Write

$$(1 + A(z-1))^{-1} = \sum_{j=-m}^{m} a_j z^j.$$

This leads to the simultaneous equations

$$\begin{aligned}
(1-A)a_{-m} &= 0 \\
(1-A)a_{-m+1} + Aa_{-m} &= 0 \\
&\vdots \\
(1-A)a_{-1} + Aa_{-2} &= 0 \\
(1-A)a_0 + Aa_{-1} &= 1 \\
(1-A)a_1 + Aa_0 &= 0 \\
&\vdots \\
(1-A)a_{m-1} + Aa_{m-2} &= 0 \\
(1-A)a_m + Aa_{m-1} &= 0 \\
Aa_m &= 0.
\end{aligned}$$

- Start from the bottom. Combine the last equation $Aa_m = 0$ with $A\times$ second-last equation, to deduce $A^2 a_{m-1} = 0$. Combine this with $A^2$ times of the third-last equation to deduce $A^3 a_{m-2} = 0$. Iteratively, we obtain

$$A^{m+1} a_0 = 0. \tag{13.10}$$

- Similarly, start from the top. Combine the first equation $(1-A)a_{-m} = 0$ with $(1-A)\times$ the second equation, to deduce $(1-A)^2 a_{-m+1} = 0$. Iteratively,
$$(1-A)^m a_{-1} = 0.$$
Combine with $(1-A)^m\times$ middle equation to get
$$(1-A)^{m+1} a_0 = (1-A)^m.$$
Multiply by $(1-A)A^{m+1}$ and combine with (13.10) to get
$$0 = (1-A)^{m+1} A^{m+1}.$$

□

The calculation below is taken from notes of J. Ebert and Rosenberg's book.

**Lemma 13.8.** *If $A \in M_n(R)$ has $A(1-A)$ nilpotent, then*
$$A = \nu + p, \qquad \text{where } p = p^2,\ \nu \text{ nilpotent},\ \nu p = p\nu.$$

*Proof.* We have $A^k(1-A)^k = 0$ for some $k$. Below, we will need the integer-coefficient polynomials $a \equiv a(t), b \equiv b(t)$, defined by the equation
$$1 = (t + (1-t))^{2k} = a(t)t^k + b(t)(1-t)^k.$$
Set
$$p := a(A)A^k \quad \Leftrightarrow \quad 1 - p = b(A)(1-A)^k.$$
Then
$$p - p^2 = p(1-p) = a(A)b(A)\underbrace{A^k(1-A)^k}_{=0} = 0,$$
so $p$ is idempotent. We also set
$$\nu := A - p,$$
and claim that $\nu$ is nilpotent and commutes with $p$.

Note that $\nu = A - a(A)A^k = f(A)$, where $f$ is the polynomial
$$f(t) = t - a(t)t^k = t(1 - a(t)t^{k-1}).$$

As $\nu$ and $p$ are both polynomials in $A$, they commute. Next, observe that

$$f(t) = t - a(t)t^k = t + b(t)(1-t)^k - 1 = (t-1)\big(1 - b(t)(1-t)^{k-1}\big)$$

has both $t-1$ and $t$ as factors, so

$$f(t) = t(t-1)g(t)$$

for some polynomial, which we check to have integer coefficients. Then we can rewrite $\nu$ as

$$\nu = f(A) = A(A-1)g(A),$$

which is nilpotent in view of $A^k(1-A)^k = 0$. □

**Theorem 13.9** (Bass–Heller–Swan; Fundamental Theorem for $K_1$)**.** *There is a natural splitting*

$$K_1(R[z^{\pm}]) \cong K_0(R) \oplus NK_{1,-}(R) \oplus NK_{1,+}(R) \oplus K_1(R).$$

*Consequently, there is a natural identification*

$$K_0(R) \cong \operatorname{coker}\Big(K_1(R[z^{-1}]) \oplus K_1(R[z]) \overset{(-\mathrm{incl}_*,\mathrm{incl}_*)}{\longrightarrow} K_1(R[z^{\pm}])\Big). \tag{13.11}$$

*Proof.* Recall the splitting of (13.1),

$$K_1(R[z^{\pm}]) \cong K_1(R[z^{\pm}], (z-1)) \oplus K_1(R).$$

Let $x \in K_1(R[z^{\pm}], (z-1))$. By Lemmas 13.7 and 13.8,

$$x = [z^{-\ell}] + [1_n + (\nu + p)(z-1)] \tag{13.12}$$

for some $\ell \in \mathbb{N}$, nilpotent $\nu$, idempotent $p$, and $p\nu = \nu p$. We have the factorization

$$\begin{aligned}
1_n + (\nu+p)(z-1) &= \big(1_n + p(z-1)\big)\Big(1_n + \big(1_n + p(z-1)\big)^{-1}\nu(z-1)\Big) \\
&= \big(1_n + p(z-1)\big)\Big(1_n + \big(pz^{-1} + (1_n - p)\big)\nu(z-1)\Big) \\
&= \big(1_n + p(z-1)\big)\Big(1_n + p\nu(1-z^{-1}) + (1_n - p)\nu(z-1)\Big) \\
&= \big(1_n + p(z-1)\big)\big(1_n - p\nu(z^{-1}-1)\big)\big(1_n + (1_n-p)\nu(z-1)\big).
\end{aligned}$$

Thus

$$\begin{aligned} x &= [z^{-\ell}] + [1_n + p(z-1)] + [1_n - p\nu(z^{-1}-1)] + [1_n + (1_n - p)\nu(z-1)] \\ &= \beta\big([pR^n] - [R^\ell]\big) + [1_n - p\nu(z^{-1}-1)] + [1_n + (1_n - p)\nu(z-1)], \end{aligned} \tag{13.13}$$

where $\beta$ is the Bott map. Note that $p\nu$ and $(1_n - p)\nu$ are nilpotent since $p$ commutes with $\nu$.

(Sketch.) In the same manner as Prop. 13.6, we may map

$$K_1(R[z^\pm], (z-1)) \overset{(\gamma,\delta_-,\delta_+)}{\longrightarrow} K_0(R) \oplus \mathrm{Nil}_0(R) \oplus \mathrm{Nil}_0(R), \tag{13.14}$$

according to the decomposition of $x$ in (13.13). Here, $\gamma$ inverts $\beta(-)$, while

$$\begin{aligned} \delta_+ : [1_n + (1_n - p)\nu(z-1)] &\mapsto [(R^n, (1_n - p)\nu)] - [(R^n, 0)] \\ \delta_- : [1_n - p\nu(z-1)] &\mapsto [(R^n, -p\nu)] - [(R^n, 0)] \end{aligned}$$

are left-inverted by

$$\begin{aligned} (\mathrm{incl.})_* \circ \tau_+ : \mathrm{Nil}_0(R) &\overset{\cong}{\to} \overbrace{NK_{1,+}(R)}^{\subseteq K_1(R[z])} \to K_1(R[z^\pm]) \\ (\mathrm{incl.})_* \circ \tau_- : \mathrm{Nil}_0(R) &\overset{\cong}{\to} \underbrace{NK_{1,-}(R)}_{\subseteq K_1(R[z^{-1}])} \to K_1(R[z^\pm]), \end{aligned}$$

respectively, where $\tau_\pm$ are isomorphisms from Prop. 13.6. So we obtain

$$K_1(R[z^\pm], (z-1)) \cong K_0(R) \oplus NK_{1,-}(R) \oplus NK_{1,+}(R),$$

thus

$$K_1(R) \cong K_1(R[z^\pm], (z-1)) \oplus K_1(R) \cong K_0(R) \oplus NK_{1,-}(R) \oplus NK_{1,+}(R) \oplus K_1(R). \tag{13.15}$$

To make (13.14) precise, we need to check a few things. If in Lemma 13.7, $\ell$ is increased, this can be absorbed via stabilization as extra copies of 1 in $A$, thus also in $p$ (for the decomposition $A = \nu + p$); the map (13.14) is unaffected. If different $A, A'$ give the same $K_1(R[z^\pm], (z-1))$-class in Lemma 13.7, related by some elementary matrices of the $1 + \mu(z-1)$ form, we have to argue like in Prop. 13.6 that the latter map to the trivial class under (13.14). Then we get a well-defined homomorphism (13.14) which is bijective.

Recall the splittings

$$K_1(R[z]) \cong NK_{1,+}(R) \oplus K_1(R),$$
$$K_1(R[z^{-1}]) \cong NK_{1,-}(R) \oplus K_1(R),$$

from (13.1). Accordingly, there is an exact sequence

$$0 \to K_1(R) \xrightarrow{\binom{\text{incl.}_*}{\text{incl.}_*}} \begin{matrix} K_1(R[z^{-1}]) \\ \oplus K_1(R[z]) \end{matrix} \xrightarrow{(-\text{incl.}_*, \text{incl.}_*)} K_1(R[z^{\pm}]) \tag{13.16}$$

with the last map landing on the $NK_{1,-}(R) \oplus NK_{1,+}(R) \oplus K_1(R)$ part of the decomposition (13.15). Then (13.11) follows. □

*Remark.* Some context: For so-called left regular rings, the $\mathrm{Nil}_0(R)$ groups are known to vanish, and we just have the splitting $K_1(R[z^{\pm}]) \cong K_0(R) \oplus K_1(R)$. This kind of results were obtained by Bass–Heller–Swan and applied to group rings $\mathbb{Z}[G]$ for $G = \mathbb{Z}^d$ (these are are of the form $R[z^{\pm}]$ with $R$ regular). A key consequence is the conclusion that the Whitehead group $\mathrm{Wh}(G)$ vanishes for $G = \mathbb{Z}^d$ (Recall Section 7.4). Applications to cobordisms and manifold topology follow along the lines of Section 7.1.

## 13.4 Negative $K$-theory

In an exercise, we showed that $K_0(-)$ may be functorially defined for non-unital rings, and coincides with the usual $K_0$ in the unital case.

In view of (13.11), we make the following definition:

**Definition 22.** For a (possibly non-unital) ring $R$, define

$$K_{-1}(R) := \mathrm{coker}\big(K_0(R[z^{-1}]) \oplus K_0(R[z])\big) \xrightarrow{\text{diff.}} K_0(R[z^{\pm}])\big),$$

with diff. denoting the difference of the morphisms induced by inclusion. We also define

$$NK_{0,+}(R) := \ker\big((\mathrm{ev}_1)_* : K_0(R[z]) \to K_0(R)\big),$$
$$NK_{0,-}(R) := \ker\big((\mathrm{ev}_1)_* : K_0(R[z^{-1}]) \to K_0(R)\big).$$

The reason for allowing non-unital $R$ is that we will need to consider $K_{-1}(I)$ for an ideal $I$ later.

**Theorem 13.10** (Fundamental Theorem for $K_0$)**.** *Let $R$ be a (possibly non-unital) ring. There is a natural splitting*

$$K_0(R[z^{\pm}]) \cong K_{-1}(R) \oplus NK_{0,-}(R) \oplus NK_{0,+}(R) \oplus K_0(R).$$

*Proof.* Start with unital $R$. In Theorem 13.9, specifically (13.16) of the proof, we may replace $R$ by $R[w^{\pm}]$, giving the 4-term exact sequence

$$0 \to K_1(R[w^{\pm}]) \to \begin{matrix} K_1(R[w^{\pm}, z^{-1}]) \\ \oplus K_1(R[w^{\pm}, z]) \end{matrix} \to K_1(R[w^{\pm}, z^{\pm}]) \leftrightarrow K_0(R[w^{\pm}]) \to 0. \tag{13.17}$$

Place this in the following diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(R) & \longrightarrow & \begin{matrix} K_0(R[z^{-1}]) \\ \oplus K_0(R[z]) \end{matrix} & \longrightarrow & K_0(R[z^{\pm}]) & \longrightarrow & K_{-1}(R) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & K_1(R[w^{\pm}]) & \longrightarrow & \begin{matrix} K_1(R[w^{\pm}, z^{-1}]) \\ \oplus K_1(R[w^{\pm}, z]) \end{matrix} & \longrightarrow & K_1(R[w^{\pm}, z^{\pm}]) & \longleftrightarrow & K_0(R[w^{\pm}]) & \longrightarrow & 0 \end{array}$$

whose first row is also exact on the right by definition of $K_{-1}$, and exact on the left because there are splittings

$$\begin{aligned} K_0(R[z]) &= NK_{0,+}(R) \oplus K_0(R), \\ K_0(R[z^{-1}]) &= NK_{0,-}(R) \oplus K_0(R). \end{aligned}$$

The vertical maps are summand inclusions $K_0(S) \hookrightarrow K_1(S[w^{\pm}])$ for $S = R, R[z^{-1}], R[z], R[z^{\pm}]$, and the diagram commutes. Chasing the summands, we deduce that the first row inherits exactness at $K_0(R[z^{-1}]) \oplus K_0(R[z])$.

It remains to show that there is a splitting $K_{-1}(R) \to K_0(R[z^{\pm}])$. We have seen that the first row shortens to an exact sequence

$$0 \to NK_{0,-}(R) \oplus NK_{0,+}(R) \oplus K_0(R) \to K_0(R[z^{\pm}]) \to K_{-1}(R) \to 0.$$

This inherits the left-split property from the second row. As we are in abelian groups, it is also right split.

For non-unital $R$, recall that $K_0(R)$ is defined as a kernel inside $K_0(\mathbb{Z} \oplus R)$; in fact, it is a direct summand. Then argue that the theorem for non-unital $R$ follows from that of the unital case by following the relevant summands carefully. □

A ring homomorphism $R \to R'$ induces homomorphisms on the various polynomial rings, and then a homomorphism on the cokernels, $K_{-1}(R) \to K_{-1}(R')$, via their splitting maps. So $K_{-1}$ is a functor from unital rings to abelian groups.

We may now iteratively define the *negative* $K$-functors.

**Definition 23.** For a (possibly non-unital) ring $R$, and $n \leq -1$, define

$$K_n(R) := \operatorname{coker}\big(K_{n+1}(R[z^{-1}]) \oplus K_{n+1}(R[z])\big) \xrightarrow{\text{diff}} K_{n+1}(R[z^{\pm}])\big), \qquad n \leq -1,$$

as well as

$$\begin{aligned} NK_{n,+}(R) &:= \ker\big((\mathrm{ev}_1)_* : K_n(R[z]) \to K_n(R)\big), \\ NK_{n,-}(R) &:= \ker\big((\mathrm{ev}_1)_* : K_n(R[z^{-1}]) \to K_n(R)\big), \qquad n \leq -1. \end{aligned}$$

Iteratively repeating the arguments of Theorem 13.10, we get the following generalization.

**Theorem 13.11** (Fundamental Theorem for negative $K$-theory)**.** *Let $R$ be a (possibly non-unital) ring. For all $n \leq 0$, there is a natural splitting*

$$K_n(R[z^{\pm}]) \cong K_{n-1}(R) \oplus NK_{n,-}(R) \oplus NK_{n,+}(R) \oplus K_n(R).$$

**Theorem 13.12.** *Let $I$ be an ideal in a unital ring $R$, and write*

$$0 \to I \xrightarrow{\iota} R \xrightarrow{q} R/I \to 0$$

*for the corresponding short exact sequence. Its $K_2$-to-$K_1$-to-$K_0$ exact sequence extends to*

$$\cdots \to K_0(R) \xrightarrow{q_*} K_0(R/I) \xrightarrow{\partial} K_{-1}(I) \xrightarrow{\iota_*} K_{-1}(R) \xrightarrow{q_*} K_{-1}(R/I) \xrightarrow{\partial} K_{-2}(I) \xrightarrow{\iota_*} \cdots,$$

*arbitrarily to the right.*

*Proof.* First, we note that the $K_1$-$K_0$ connecting map (denoted $\partial$) is natural with respect to the short exact sequence, as may be deduced from its explicit formula. We combine this with the 4-term split exact sequences from the

Fundamental Theorems, and functoriality of $K_1, K_0$, to get a commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1(R) & \longrightarrow & \begin{array}{c}K_1(R[z^{-1}])\\ \oplus K_1(R[z])\end{array} & \longrightarrow & K_1(R[z^{\pm}]) & \longleftrightarrow & K_0(R) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \\
0 & \longrightarrow & K_1(R/I) & \longrightarrow & \begin{array}{c}K_1((R/I)[z^{-1}])\\ \oplus K_1((R/I)[z])\end{array} & \longrightarrow & K_1((R/I)[z^{\pm}]) & \longleftrightarrow & K_0(R/I) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \vdots{\scriptstyle \partial} & & \\
0 & \longrightarrow & K_0(I) & \longrightarrow & \begin{array}{c}K_0(I[z^{-1}])\\ \oplus K_0(I[z])\end{array} & \longrightarrow & K_0(I[z^{\pm}]) & \longleftrightarrow & K_{-1}(I) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \\
0 & \longrightarrow & K_0(R) & \longrightarrow & \begin{array}{c}K_0(R[z^{-1}])\\ \oplus K_0(R[z])\end{array} & \longrightarrow & K_0(R[z^{\pm}]) & \longleftrightarrow & K_{-1}(R) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \\
0 & \longrightarrow & K_0(R/I) & \longrightarrow & \begin{array}{c}K_0((R/I)[z^{-1}])\\ \oplus K_0((R/I)[z])\end{array} & \longrightarrow & K_0((R/I)[z^{\pm}]) & \longleftrightarrow & K_{-1}(R/I) & \longrightarrow & 0
\end{array}.$$

The first three columns are exact. The splittings give us the dotted $\partial : K_0(R/I) \to K_{-1}(I)$ arrow making the last column exact, and we can patch this with the first column to get an exact sequence until $K_{-1}(R/I)$.

Check that $\partial : K_0(R/I) \to K_{-1}(I)$ is also natural (Exercise). Then, because of the fundamental theorem for negative $K$-theory, we can repeat the argument for the same diagram but with all indices decreased by one. This yields the $\partial : K_{-1}(R/I) \to K_{-2}(I)$ map which extends the exact sequence until $K_{-2}(R/I)$. This argument can be iterated indefinitely. $\square$