2 Yoneda

2.1 Yoneda Lemma and embedding

Yoneda embedding is the functorial elaboration of a "physically obvious" idea:

An object is studied by probing that object.

Specifically, each object $c \in \mathrm{Ob}(\mathcal{C})$ is associated with a *contravariant* hom-functor,

$$h_c := \mathrm{Hom}_{\mathcal{C}}(-, c) \in \mathrm{Ob}(\mathbf{Set}_{\mathcal{C}}).$$

By definition, h_c converts each morphism $c_1 \xrightarrow{f} c_2$ in $\mathcal{C}$ into a pull-back map

$$\begin{array}{ccc} h_c(c_1) & \overset{h_c(f)=f^*}{\longleftarrow} & h_c(c_2) \\ \left(c_1 \xrightarrow{a\circ f} c\right) & \longleftarrow\!\shortmid & \left(c_2 \xrightarrow{a} c\right). \end{array}$$

The assignment of c to h_c is the object-level part of a *functor*,

$$\begin{aligned} h_{(-)} : \mathcal{C} &\to \mathbf{Set}_{\mathcal{C}} \\ c &\mapsto h_c \\ \left(c \xrightarrow{a} c')\right) &\mapsto \left(h_c \overset{a_*}{\Rightarrow} h_{c'}\right). \end{aligned} \tag{2.1}$$

Above, a_* denotes pushforward-by a (i.e., post-composition with a), and it does define a natural transformation, because for every morphism $c_1 \xrightarrow{f} c_2$ in $\mathcal{C}$, the following commutes,

$$\begin{array}{ccc} h_c(c_1) & \xleftarrow{h_c(f)\equiv f^*} & h_c(c_2) \\ \downarrow{\scriptstyle a_*} & & \downarrow{\scriptstyle a_*} \\ h_{c'}(c_1) & \xleftarrow{h_{c'}(f)\equiv f^*} & h_{c'}(c_2) \end{array} .$$

(Obviously push-forward commutes with pullback.)

For historical reasons, a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is called a *presheaf* on $\mathcal{C}$. The functor $h_{(-)}$ (Eq. (2.1)) is called the *Yoneda embedding*, turning objects/morphisms of $\mathcal{C}$ into presheafs/natural transformations.

If a presheaf $F \in \mathrm{Ob}(\mathbf{Set}_{\mathcal{C}})$ is naturally isomorphic to some h_c, we say that F is *representable* by the object $c \in \mathrm{Ob}(\mathcal{C})$. In general, there exist non-representable presheafs.

Understanding relationships (morphisms) between objects in a category $\mathcal{C}$ is a key motivation for using categorical language in the first place. So, for the Yoneda embedding of $\mathcal{C}$ into $\mathbf{Set}_{\mathcal{C}}$ to be useful, we should make sure that these relationships do not get lost. Indeed, we will see that the hom-sets

$$\mathrm{Hom}_{\mathcal{C}}(c, c'), \qquad \mathrm{Hom}_{\mathbf{Set}_{\mathcal{C}}}(h_c, h_{c'}) = \mathrm{Nat}(h_c, h_{c'})$$

are naturally identified under the Yoneda embedding.

The following is the famous basic Lemma underpinning the Yoneda embedding concept.

Lemma 2.1 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a (locally small) category. For all $c \in \mathrm{Ob}(\mathcal{C})$ and $F \in \mathrm{Ob}(\mathbf{Set}_{\mathcal{C}})$, the "initial-value" maps*

$$\begin{aligned} I_{c,F} : \mathrm{Nat}(h_c, F) &\to F(c), \qquad c \in \mathrm{Ob}(\mathcal{C}) \\ \eta &\mapsto \eta_c(\mathrm{id}_c), \end{aligned} \tag{2.2}$$

are bijections.

Proof. Fix $c \in \mathrm{Ob}(\mathcal{C})$. By definition, a natural transformation $\eta : h_c \Rightarrow F$ is a collection of (set-theoretic) functions

$$\eta_{c_1} : h_c(c_1) \to F(c_1), \qquad c_1 \in \mathrm{Obj}(\mathcal{C}),$$

making the following commute,

$$\begin{array}{ccc} h_c(c_1) & \xleftarrow{\ f^*\ } & h_c(c_2) \\ \Big\downarrow{\scriptstyle \eta_{c_2}} & & \Big\downarrow{\scriptstyle \eta_{c_2}} \\ F(c_1) & \xleftarrow{\ F(f)\ } & F(c_2) \end{array}, \qquad \forall\, f \in \mathrm{Hom}_{\mathcal{C}}(c_1, c_2). \tag{2.3}$$

In particular, setting $c_2 = c$ and starting with $\mathrm{id}_c \in h_c(c) \equiv \mathrm{Hom}_{\mathcal{C}}(c, c)$ at the top right,

$$\begin{array}{ccc} f & \xleftarrow{\ f^*\ } & \mathrm{id}_c \\ \Big\downarrow{\scriptstyle \eta_{c_1}} & & \Big\downarrow{\scriptstyle \eta_c} \\ \eta_{c_1}(f) & \xleftarrow{\ F(f)\ } & \eta_c(\mathrm{id}_c) \end{array}, \qquad \forall\, f \in \mathrm{Hom}_{\mathcal{C}}(c_1, c), \tag{2.4}$$

commutes. Eq. (2.4) shows that a natural transformation $\eta : h_c \Rightarrow F$ is completely specified by its "initial value" $\eta_c(\mathrm{id}_c)$, via the formula

$$\eta_{c_1}(f) = F(f)(\eta_c(\mathrm{id}_c)), \qquad f \in \mathrm{Hom}_{\mathcal{C}}(c_1, c) = h_c(c_1). \tag{2.5}$$

So two natural transformations $h_c \Rightarrow F$ with the same initial value must be exactly the same natural transformation. This shows that $I_{c,F}$ (Eq. (2.2)) is injective.

For surjectivity of $I_{c,F}$, start with $x \in F(c)$. We need to find a natural transformation $\eta : h_c \Rightarrow F$ solving $I_{c,F}(\eta) = \eta_c(\mathrm{id}_c) = x$. Guided by (2.5), we define the functions

$$\begin{aligned} \eta_{c_1} : h_c(c_1) &\to F(c_1) \\ f &\mapsto F(f)(x), \qquad c_1 \in \mathrm{Obj}(\mathcal{C}). \end{aligned} \tag{2.6}$$

and verify that $\eta \equiv \{\eta_{c_1}\}_{c_1 \in \mathrm{Ob}(\mathcal{C})}$ is a natural transformation. So we need to check that

$$\begin{array}{ccc} h_c(c_1) & \xleftarrow{h_c(g)=g^*} & h_c(c_2) \\ \downarrow{\scriptstyle \eta_{c_1}} & & \downarrow{\scriptstyle \eta_{c_2}} \\ F(c_1) & \xleftarrow{F(g)} & F(c_2) \end{array}, \qquad \forall\, g \in \mathrm{Hom}_{\mathcal{C}}(c_1, c_2),$$

is a commuting diagram in **Set**. Indeed, starting from any element $f \in \mathrm{Hom}_{\mathcal{C}}(c_2, c)$ in the top right, and using the definition (2.6) for η, we have the diagram

$$\begin{array}{ccc} f \circ g & \xleftarrow{\quad g^* \quad} & f \\ \downarrow{\scriptstyle \eta_{c_1}} & & \downarrow{\scriptstyle \eta_{c_2}} \\ \underbrace{\eta_{c_1}(f \circ g)}_{F(f\circ g)(x)} & \xleftarrow{\quad F(g) \quad} & \underbrace{\eta_{c_2}(f)}_{F(f)(x)} \end{array},$$

which commutes because F is a contravariant functor (thus $F(f \circ g) = F(g) \circ F(f)$).

Since η is a natural transformation, we may now calculate that

$$I_{c,F}(\eta) = \eta_c(\mathrm{id}_c) = F(\mathrm{id}_c)(x) = \mathrm{id}_{F(c)}(x) = x,$$

as required. □

Exercise 2.1. Show that the isomorphisms given in the Yoneda Lemma vary naturally with respect to c and F.

Corollary 2.2 (Yoneda embedding)**.** *For any (locally small) category $\mathcal{C}$, the Yoneda embedding functor* (2.1) *is fully faithful, i.e., the assignments*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(c, c') &\to \mathrm{Nat}(h_c, h_{c'}) \equiv \mathrm{Hom}_{\mathbf{Set}_{\mathcal{C}}}(h_c, h_{c'}) \\ a &\mapsto a_* \end{aligned}$$

are bijections.

Proof. Set $F = h_{c'}$ in the Yoneda Lemma 2.1, to get bijections

$$I_{c,h_{c'}} : \mathrm{Nat}(h_c, h_{c'}) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(c, c'), \qquad \forall\, c \in \mathrm{Ob}(\mathcal{C}).$$

Let $a \in \mathrm{Hom}_{\mathcal{C}}(c, c')$. Substituting $F = h_{c'}$ into the general formula (2.6), $I^{-1}_{c,h_{c'}}(a)$ is the natural transformation

$$\begin{aligned} I^{-1}_{c,h_{c'}}(a)_{c_1} : h_c(c_1) &\to h_{c'}(c_1) \\ f &\mapsto h_{c'}(f)(a) = f^*(a) = a_*(f), \qquad c_1 \in \mathrm{Ob}(\mathcal{C}). \end{aligned}$$

In other words, the bijection $I^{-1}_{c,h_{c'}}$ is precisely the morphism assignment $a \mapsto a_*$ of the Yoneda embedding. □

Remark 2. Suppose c, c' are isomorphic, then because the Yoneda embedding is a functor, the presheafs $h_c, h_{c'}$ will be naturally isomorphic. Conversely, suppose $h_c, h_{c'}$ are naturally isomorphic. Because the Yoneda embedding is fully faithful (bijective on hom-sets), it follows readily that c, c' must be isomorphic.

3 Simplicial sets

The *simplex category* Δ is the small category with:

- Objects being the ordered sets $[n] = \{0, \ldots, n\}$, $n \geq 0$;
- Morphisms being order-preserving functions.

We may Yoneda embed Δ into $\mathbf{Set}_\Delta$,

$$\Delta^n := h_{[n]} \equiv \mathrm{Hom}_\Delta(-, [n]) \in \mathrm{Ob}(\mathbf{Set}_\Delta).$$

These Δ^n are the representable objects of $\mathbf{Set}_\Delta$, and are called the *standard simplicial sets*.

General objects of $\mathbf{Set}_\Delta$ are called *simplicial sets*[2]. As we will learn, they are all built up from the standard Δ^n by some "limiting" process. This is closely related to ideas from algebraic topology, where one studies how topological spaces are built up from topological simplices (points, lines, triangles,...). Indeed, this analogy is precise at the level of an adjunction relating hom-sets of **Top** and $\mathbf{Set}_\Delta$. So there is a sense in which simplicial sets serve as a kind of "generalized topological space".

Here is a general class of simplicial sets which do not immediately look like topological spaces.

Example 3.1. Any small category $\mathcal{C}$ determines a simplicial set, called the *nerve* of $\mathcal{C}$, and denoted $N(\mathcal{C})$. At the object level, $N(\mathcal{C})([n])$ is the set of n-arrow-compositions,

$$N(\mathcal{C})([n]) := \left\{ X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots X_{n-1} \xrightarrow{f_n} X_n \right\}, \qquad [n] \in \mathrm{Ob}(\Delta).$$

As for the morphism level assignment, note that given an order-preserving map $a \in \mathrm{Hom}_\Delta([m], [n])$, we can map any n-arrow-composition to an m-arrow-composition

$$X_{a(0)} \to X_{a(1)} \to \cdots \to X_{a(m)},$$

by suitably combining the f_i and/or inserting some extra identity arrows. This defines a function

$$N(\mathcal{C})(a) : N(\mathcal{C})([n]) \to N(\mathcal{C})([m]).$$

[2]Condusingly, a simplicial set is not itself a set, but is a set-valued presheaf on Δ.

It follows that $N(\mathcal{C})$ is a functor $\Delta^{\mathrm{op}} \to \mathbf{Set}$.

In fact, if we think of $[n]$ as a category with objects $0, 1, \ldots, n$ and arrows pointing in non-decreasing directions, then $N([n])$ is precisely Δ^n. (Exercise.) But not every simplicial set arises as the nerve of some category.

Next, we see how simplicial sets arise naturally from topological spaces.

3.1 Singular homology

The *standard topological n-simplex*, denoted $|\Delta^n|$, is the convex hull of the standard basis vectors $e_0, \ldots, e_n$ for $\mathbb{R}^{n+1}$,

$$|\Delta^n| = \{(t_0, \ldots, t_n) \,:\, t_i \geq 0,\, t_0 + \cdots + t_n = 1\}.$$

We use the standard $|\Delta^n|$ as an "n-dimensional probe" to investigate general topological spaces X. As we will see, $|\Delta^n|$ is the "geometric realization" of Δ^n, which explains the notation.

Let X be a topological space. Continuous maps $\sigma : |\Delta^n| \to X$ are called *singular n-simplices* in X, and they form the set

$$\mathrm{Sing}_n(X) := \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X).$$

- As $|\Delta^0| = \{e_0\}$ is a one-point set, $\mathrm{Sing}_0(X)$ is the set of points in X.
- $\mathrm{Sing}_1(X)$ is the set of continuous (unparametrized) paths in X.

The sets $\mathrm{Sing}_n(X), n \geq 0$ are not independent of each other. For example, $|\Delta^n|$ has $n+1$ *faces*, where the i-th face is defined as the subset with $t_i = 0$. The face-inclusion maps are

$$\begin{aligned} d^i : |\Delta^{n-1}| &\xrightarrow{d^i} |\Delta^n| \\ (t_0, \ldots, t_{n-1}) &\mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}). \end{aligned} \tag{3.1}$$

Therefore, a singular n-simplex $\sigma : |\Delta^n| \to X$ automatically determines the singular $(n-1)$-simplices $\sigma \circ d^i$, $i = 0, \ldots, n$. Then the *boundary* of $\sigma \in \mathrm{Sing}_n(X)$ can be defined as the formal alternating sum of its face singular $(n-1)$-simplices,

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i (\sigma \circ d^i).$$

For example, for $\sigma \in \mathrm{Sing}_1(X)$, we have

$$(\partial\sigma)(e_0) = \sigma(e_1) - \sigma(e_0),$$

expressing the statement that "the boundary of a path is its endpoint minus its starting point".

The *singular homology* functors of algebraic topology are obtained by $\mathbb{Z}$-linearly exploiting the principle that $\partial \circ \partial = 0$. In more detail:

- For each $n \geq 0$, let $C_n(X)$ be the free abelian group generated by $\mathrm{Sing}_n(X)$. So elements of $C_n(X)$ are formal $\mathbb{Z}$-linear combinations of singular n-simplices, called *singular n-chains* in X.

  For example, $\partial\sigma \in C_{n-1}(X)$ is a singular $(n-1)$-chain.

- The boundary operation $\partial : \mathrm{Sing}_n(X) \to C_{n-1}(X)$ extends $\mathbb{Z}$-linearly to abelian group homomorphisms

  $$\partial : C_n(X) \to C_{n-1}(X), \qquad n \geq 1.$$

  Some combinatorics lead to the crucial property,

  $$\partial \circ \partial = 0.$$

- Combine the ∂ maps for various n into a sequence of abelian group homomorphisms

  $$\cdots \xrightarrow{\partial} C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(X) \xrightarrow{\partial} 0. \qquad (3.2)$$

  Eq. (3.2) is the *singular chain complex* $C_\bullet(X)$ of X.

- Since $\partial \circ \partial = 0$, the quotient abelian groups

  $$H_n(X) = \frac{\ker\big(\partial : C_n(X) \to C_{n-1}(X)\big)}{\mathrm{Im}\big(\partial : C_{n+1}(X) \to C_n(X)\big)}, \qquad n \geq 0,$$

  are well defined, and they are called the *singular homology groups* of X.

Sketch of singular homology functors. The graded abelian group $H_\bullet(X) = \{H_n(X)\}_{n\geq 0}$ depends only on the chain complexes (3.2) rather than the sets $\mathrm{Sing}_n(X)$ themselves. Abstractly, chain complexes (of $\mathbb{Z}$-modules) form a category $\mathbf{Ch}_\mathbb{Z}$ (with "chain maps" as morphisms), so the singular chain complex $C_\bullet(X)$ can be understood as a particular object in this category, which happens to be built from a topological space. Indeed, homology groups make sense for any chain complex.

When $f : X \to Y$ is a morphism in **Top**, we get a chain map from $C_\bullet(X)$ to $C_\bullet(Y)$, and a standard "diagram-chasing" leads to induced group homomorphisms $H_n(f) : H_n(X) \to H_n(Y)$, $n \geq 0$. So the $H_n(-)$ are actually functors $\mathbf{Top} \to \mathbf{Ab}$. It is perhaps better to view $H_\bullet(-)$ as a functor $\mathbf{Top} \to \mathbf{Ch}_\mathbb{Z} \to \mathbf{GrAb}$ to graded abelian groups, factoring through $\mathbf{Ch}_\mathbb{Z}$.

3.1.1 Simplicial set viewpoint

Since the list of sets $\mathrm{Sing}(X) = \{\mathrm{Sing}_n(X)\}_{n\geq 0}$ prefigures the chain complex $C_\bullet(X)$, there is more information in $\mathrm{Sing}(X)$. But it is not very obvious what kind of object $\mathrm{Sing}(X)$ is, or what morphisms $\mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ are. Below, we shall explain why $\mathrm{Sing}(X)$ is a simplicial set in a natural way.

First, observe that we can affinely place $|\Delta^m|$ into $|\Delta^n|$ by specifying a map from the ordered basis $\{e_0, \ldots, e_m\}$ to the ordered basis $\{e_0, \ldots, e_n\}$. Equivalently, any function a from the ordered set $[m] = \{0, \ldots, m\}$ to $[n] = \{0, \ldots, n\}$ induces the placement

$$\begin{aligned} \tilde{a} : |\Delta^m| &\to |\Delta^n| \\ t = (t_0, \ldots, t_m) &\mapsto \Big(\sum_{i\in a^{-1}(0)} t_i, \ldots, \sum_{i\in a^{-1}(n)} t_i \Big). \end{aligned} \tag{3.3}$$

When two such functions a_1, a_2 are composable, we have

$$\widetilde{a_2 \circ a_1} = \widetilde{a_2} \circ \widetilde{a_1}.$$

For example, for $0 \leq i \leq n$:

- If $d^i : [n-1] \to [n]$ is the injective map which misses i, then $\widetilde{d^i} : |\Delta^{n-1}| \to |\Delta^n|$ is the i-th face-inclusion, see (3.1).
- If $s^i : [n+1] \to [n]$ is the surjective map which repeats i, then $\widetilde{s^i} : |\Delta^{n+1}| \to |\Delta^n|$ is the map which collapses the edge $\{(t_0, \ldots, t_n) : t_i + t_{i+1} = 1\}$ to the vertex e_i.

By the above discussion, particularly (3.3) applied to the order-preserving morphisms in Δ, we have a *functor*

$$\begin{aligned} \Delta &\to \mathbf{Top} \\ [n] &\mapsto |\Delta^n| \\ \Big([m] \xrightarrow{a} [n]\Big) &\mapsto \Big(|\Delta^m| \xrightarrow{\tilde{a}} |\Delta^n|\Big). \end{aligned} \tag{3.4}$$

Further composing with the contravariant hom-functor $\mathrm{Hom}_{\mathbf{Top}}(-, X)$, we get the contravariant functor

$$\begin{aligned} \mathrm{Sing}(X) : \Delta &\to \mathbf{Set} \\ [n] &\mapsto \mathrm{Sing}_n(X) \\ \Big([m] \xrightarrow{a} [n]\Big) &\mapsto \Big(\mathrm{Sing}_n(X) \xrightarrow{\tilde{a}^*} \mathrm{Sing}_m(X)\Big). \end{aligned}$$

We conclude that $\mathrm{Sing}(X)$ *is a simplicial set.*

As $\mathrm{Sing}(-)$ varies with the choice of topological space, we can further view it as a *functor*,

$$\begin{aligned} \mathrm{Sing} : \mathbf{Top} &\to \mathbf{Set}_\Delta \\ X &\mapsto \mathrm{Sing}(X) \\ \Big(f \in \mathrm{Hom}_{\mathbf{Top}}(X, Y)\Big) &\mapsto \Big(f_* \in \mathrm{Nat}\big(\mathrm{Sing}(X), \mathrm{Sing}(Y)\big)\Big). \end{aligned} \tag{3.5}$$

Here, f_* denotes the "push-forward" natural transformation $\mathrm{Sing}(X) \Rightarrow \mathrm{Sing}(Y)$. In more detail, f_* refers to the collection of push-forward morphisms

$$\begin{aligned} f_{*,[n]} : \underbrace{\mathrm{Sing}(X)([n])}_{\mathrm{Sing}_n(X)} &\to \underbrace{\mathrm{Sing}(Y)([n])}_{\mathrm{Sing}_m(X)}, \qquad [n] \in \mathrm{Ob}(\Delta), \\ \sigma &\mapsto f \circ \sigma, \end{aligned}$$

which are such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Sing}(X)([n]) & \xrightarrow{\mathrm{Sing}(X)(a)=\tilde{a}^*} & \mathrm{Sing}(X)([m]) \\ \Big\downarrow{\scriptstyle f_{*,[n]}} & & \Big\downarrow{\scriptstyle f_{*,[m]}} \\ \mathrm{Sing}(Y)([n]) & \xrightarrow{\mathrm{Sing}(Y)(a)=\tilde{a}^*} & \mathrm{Sing}(Y)([m]) \end{array} \quad , \qquad \forall\, [m] \xrightarrow{a} [n].$$

(Because push-forward and pullback commute.)

3.2 Yoneda for simplicial sets

Corollary 3.1. *For any simplicial set S, there are canonical bijections*

$$S([n]) \cong \mathrm{Nat}(\Delta^n, S), \qquad \forall\, [n] \in \mathrm{Ob}(\Delta). \tag{3.6}$$

Proof. Since $\Delta^n \equiv h_{[n]}$, this is a special case of the *Yoneda Lemma* 2.1. □

In particular, for the simplicial set $\mathrm{Sing}(X)$ associated to a topological space, Corollary 3.1 says that there are canonical bijections

$$\mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X) \equiv \mathrm{Sing}_n(X) \cong \mathrm{Nat}(\Delta^n, \mathrm{Sing}(X)) \equiv \mathrm{Hom}_{\mathbf{Set}_\Delta}(\Delta^n, \mathrm{Sing}(X)). \tag{3.7}$$

Eq. (3.7) looks very much like an adjunction relation, except that we only know what $|\cdot|$ does to Δ^n. Later, we will explain how to define $|\cdot|$ for arbitrary simplicial sets S.

Traditionally, in the category **Top**, we use the standard topological simplices $|\Delta^n|$ to probe general topological spaces X. Elements of $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathbf{Top}}(|\Delta_n|, X)$ were called singular n-simplices in X. Likewise, in $\mathbf{Set}_\Delta$, we use the standard simplicial sets Δ^n to probe general simplicial sets S. Elements of $\mathrm{Nat}(\Delta^n, S) = \mathrm{Hom}_{\mathbf{Set}_\Delta}(\Delta^n, S)$ are called *abstract n-simplices in S*, and they are Yoneda-identified with elements of $S([n])$ by Corollary 3.1.