7 Fredholm index and Toeplitz index theorem

The counting of a formal difference $[p]-[q]$ of finite-rank projections by their "virtual rank" occurs in the classical concept of Fredholm index. In other words, $K_0(\mathcal{K})\cong\mathbb{Z}$ occurs naturally as an analytically-defined invariant for the class of Fredholm operators. As the abstract machinery needed to explain this K-theoretic viewpoint might seem unmotivated at first glance, let us instead spend some time understanding Fredholm operators, which are rather concrete things. We will also prove the oldest index theorem in the world. This will show how analytic and topological notions come together in the concept of index, and provide motivation for subsequent K-theory constructions.

We should also mention that Fredholm operators arise very naturally when considering certain PDEs, particularly elliptic ones on compact manifolds.

The classical Fredholm index

- If $T:V_0\to V_1$ is a linear operator between *finite dimensional* vector spaces, there is an exact sequence of vector spaces

  $$0\to\ker(T)\to V_0\xrightarrow{T}V_1\to\operatorname{coker}(T)\to 0,$$

  where $\operatorname{coker}(T)=V_1/\operatorname{Range}(T)$. In linear algebra, we learn that

  $$\underbrace{\dim\ker(T)}_{\text{nullity}}-\dim V_0+\underbrace{\dim V_1-\dim\operatorname{coker}(T)}_{\text{rank}}=0.$$

  Thus the *index*

  $$\operatorname{Ind}(T):=\dim\ker(T)-\dim\operatorname{coker}(T)=\dim(V_0)-\dim(V_1),\qquad(7.1)$$

  does not depend on the operator T.

- Now let $T\in\mathcal{B}(\mathcal{H}_0,\mathcal{H}_1)$ be a bounded operator between Hilbert spaces. We can restrict attention to *Fredholm operators*, which by definition, are operators such that

  $$\operatorname{Range}(T)\text{ closed},\qquad\dim\ker(T)<\infty,\qquad\dim\operatorname{coker}(T)<\infty.\qquad(7.2)$$

We write $\mathcal{F}(\mathcal{H}_0, \mathcal{H}_1) \subseteq \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$ for the subset of Fredholm operators. The *Fredholm index* of a Fredholm operator is defined to be

$$\begin{aligned} \operatorname{Ind}(T) &:= \dim\ker(T) - \dim\operatorname{coker}(T) \\ &= \dim\ker(T) - \dim\ker(T^*) \ \in \mathbb{Z}, \qquad T \in \mathcal{F}(\mathcal{H}_0, \mathcal{H}_1). \end{aligned} \tag{7.3}$$

For the second formula for the index, we used the closed range[16] condition to write

$$\operatorname{coker}(T) = \mathcal{H}_1/\operatorname{Range}(T) \cong \operatorname{Range}(T)^\perp = \ker T^*.$$

If $\mathcal{H}_0, \mathcal{H}_1$ are infinite-dimensional, then in analogy to (7.1), the Fredholm index is formally equal to

$$\operatorname{Ind}(T) = \text{“}\dim(\mathcal{H}_0) - \dim(\mathcal{H}_1)\text{”} = \infty - \infty.$$

Of course, the right side is ill-defined in a literal sense.

Instead, what a Fredholm operator T does is to identify some *sub*space of $\mathcal{H}_0$ with some *sub*space of $\mathcal{H}_1$, namely,

$$(\ker(T))^\perp \cong \operatorname{Range}(T).$$

We declare that the infinite dimensions of these subspaces “cancel out”. What remains are *finite* dimensional subspaces $\ker(T) \subseteq \mathcal{H}_0$ and $\ker(T^*) \subseteq \mathcal{H}_1$, and the Fredholm index of T measures the difference in their dimensions.

- **Proposition:** The composition of Fredholm operators is Fredholm, and the adjoint of a Fredholm operator is Fredholm. Furthermore,

$$\operatorname{Ind}(T_1T_2) = \operatorname{Ind}(T_1) + \operatorname{Ind}(T_2), \tag{7.4}$$

$$\operatorname{Ind}(T^*) = -\operatorname{Ind}(T). \tag{7.5}$$

Proof: Quite generally, there is an exact sequence

$$0 \to \ker(T_2) \to \ker(T_1T_2) \to \ker(T_1) \to \operatorname{coker}(T_2) \to \operatorname{coker}(T_1T_2) \to \operatorname{coker}(T_1) \to 0$$

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[16] The closed range condition in the definition of Fredholmness, (7.2), is actually redundant. To see this, consider T as an injective map $\ker(T)^\perp \to \mathcal{H}$. Although we do not know (yet) that $\operatorname{Range}(T)$ is closed, it has a complementary subspace of finite dimension (say n), so we can extend T to a bijective continuous map $\tilde{T} : \ker(T)^\perp \oplus \mathbb{C}^n \to \mathcal{H}$. By the bounded inverse theorem, $\tilde{T}$ is a homeomorphism, so it actually restricts to a homeomorphism $\ker(T)^\perp \overset{\cong}{\to} \operatorname{Range}(T)$. As $\ker(T)^\perp$ is closed, so is $\operatorname{Range}(T)$.

of vector spaces. By assumption, $\ker(T_i)$ and $\operatorname{coker}(T_i)$ are finite-dimensional, so the same is true for T_1T_2 by exactness. In general, the alternating sum of dimensions in an exact sequence of finite-dimensional vector spaces is zero (Exercise), so the formula (7.4) follows.

A Fredholm T has closed range, thus T^* has closed range as well (Exercise). Then we have

$$\ker(T^*) = \underbrace{\mathrm{Range}(T)^\perp}_{\cong \operatorname{coker}(T)}, \qquad \underbrace{\mathrm{Range}(T^*)^\perp}_{\cong \operatorname{coker}(T^*)} = \ker(T),$$

and formula (7.5) follows. $\square$

- **Theorem:** (Atkinson.) An operator $T \in \mathcal{B}(\mathcal{H})$ is Fredholm iff T is invertible modulo compact operators.

Proof: ($\Rightarrow$) Let T be a Fredholm operator on $\mathcal{H}$. We can restrict it to an invertible operator

$$\check{T} : \ker(T)^\perp \to \mathrm{Range}(T)$$

between Hilbert subspaces of $\mathcal{H}$. Let $S : \mathrm{Range}(T) \to \ker(T)^\perp$ be the (bounded) inverse of $\check{T}$, and extend it to an operator $S \in \mathcal{B}(\mathcal{H})$ by setting it to be zero on $\mathrm{Range}(T)^\perp$. Then we have

$$\begin{aligned} 1 - ST &= P_{\ker(T)} \\ 1 - TS &= P_{\mathrm{Range}(T)^\perp}, \end{aligned} \tag{7.6}$$

where $P_{(\cdot)}$ denotes the orthogonal projection onto $(\cdot)$. By the Fredholm assumption, the projections $1 - ST$ and $1 - TS$ are finite-rank, thus compact. So S is an inverse to T modulo compacts.

($\Leftarrow$) Suppose there exists $S \in \mathcal{B}(\mathcal{H})$ such that

$$K_1 = 1 - ST \in \mathcal{K}(\mathcal{H}), \qquad K_2 = 1 - TS \in \mathcal{K}(\mathcal{H}).$$

As $\mathcal{K}$ is the closure of finite-rank operators, we have finite-rank F_1, F_2 such that $\|K_1 - F_1\| < 1$ and $\|K_2 - F_2\| < 1$. As $\mathcal{B}(\mathcal{H})$ is a Banach algebra, we have invertibility of

$$1 + F_1 - K_1, \qquad 1 + F_2 - K_2.$$

Now calculate

$$(1+F_1-K_1)^{-1}ST = (1+F_1-K_1)^{-1}(1-K_1) = 1-\underbrace{(1+F_1-K_1)^{-1}F_1}_{\text{finite-rank}}.$$

This means that T is left-invertible up to some finite-rank operator, and it follows that $\dim\ker(T)$ is finite. Similarly,

$$TS(1+F_2-K_2)^{-1} = (1-K_2)(1+F_2-K_2)^{-1} = 1-\underbrace{F_2(1+F_2-K_2)^{-1}}_{\text{finite-rank}}$$

shows that T is right-invertible up to some finite-rank operator, thus it has $\dim\operatorname{coker}(T)$ finite. The closed range condition is automatic (Footnote 16). $\square$

(Remark.) The proof shows that T is Fredholm iff it is invertible modulo finite-rank operators.

Calkin algebra and stability of index

- Recall that $\mathcal{K}$ is a closed ideal in $\mathcal{B}$. (We suppress explicit reference to the Hilbert space $\mathcal{H}$ for ease of notation.) So there is a SES of C^*-algebras,

$$0 \to \mathcal{K} \to \mathcal{B} \xrightarrow{\pi} \underbrace{\mathcal{B}/\mathcal{K}}_{\mathcal{Q}} \to 0. \tag{7.7}$$

  The unital quotient algebra $\mathcal{Q} = \mathcal{B}/\mathcal{K}$ is called the *Calkin algebra*.

  Atkinson's theorem can be rephrased as:

$$T \text{ Fredholm } \Leftrightarrow \pi(T) \text{ invertible in } \mathcal{Q}. \tag{7.8}$$

- **Proposition:** $\mathcal{F}$ is an open subset of $\mathcal{B}$, and $\operatorname{Ind} : \mathcal{F} \to \mathbb{Z}$ is continuous (thus locally constant). Furthermore, if T is Fredholm and K is compact, then $T+K$ is Fredholm, with

$$\operatorname{Ind}(T+K) = \operatorname{Ind}(T). \tag{7.9}$$

  Proof: $\mathcal{Q}^\times$ is open because $\mathcal{Q}$ is a C^*-algebra (thus Banach algebra). By Atkinson's theorem, (7.8), $\mathcal{F} = \pi^{-1}(\mathcal{Q}^\times)$, so $\mathcal{F}$ is open. The stability of the Fredholm condition under compact perturbation is also a consequence of Atkinson's theorem.

Let $j : \ker(T)^{\perp} \to \mathcal{H}$ be the inclusion and $q : \mathcal{H} \to \operatorname{Range}(T)$ be the orthogonal projection. They are Fredholm operators, with

$$\operatorname{Ind}(j) = -\dim\ker(T), \qquad \operatorname{Ind}(q) = \dim\operatorname{coker}(T),$$

thus

$$\operatorname{Ind}(T) + \operatorname{Ind}(j) + \operatorname{Ind}(q) = 0.$$

Now, $qTj : \ker(T)^{\perp} \to \operatorname{Range}(T)$ is an invertible operator. So for all T' sufficiently close to T in operator norm, T' is Fredholm with $qT'j$ remaining invertible. So $qT'j$ is Fredholm with index zero, and by additivity of index, (7.4),

$$0 = \operatorname{Ind}(j) + \operatorname{Ind}(T') + \operatorname{Ind}(q). \tag{7.10}$$

We have shown that $\operatorname{Ind}(T') = \operatorname{Ind}(T)$ for all (Fredholm) T' which are sufficiently close to T. This shows that the index map is continuous.

Finally, (7.9) follows by considering the norm-continuous path $t \mapsto T + tK$ of Fredholm operators, and the continuity of index (thus constancy of index along the path). ☐

- Note that $\dim\ker T$ and $\dim\ker T^*$ can jump *discontinuously*, when we continuously vary the Fredholm operator T. For example, consider $1 - tp_0$, where p_0 is a rank-1 projection. It is a remarkable fact that despite these instabilities, the difference in dimensions (i.e., the index) is stable.

Toeplitz index theorem

- Let S^1 be the unit circle in $\mathbb{C}$. We may label its points by $z = e^{i\theta}, \theta \in \mathbb{R}/(2\pi\mathbb{Z})$. With the standard measure $d\theta$, we have the Hilbert space $L^2(S^1) \equiv L^2(S^1, d\theta)$.

  The *Hardy space* $H^2(S^1)$ is the Hilbert subspace of $L^2(S^1)$ spanned by $\{\psi_n : n \geq 0\}$, where

  $$\psi_n(z) = z^n.$$

  By Fourier theory[17], $L^2(S^1)$ has orthogonal basis $\{\psi_n : n \in \mathbb{Z}\}$. More specifically, the Fourier transform is a unitary isomorphism $L^2(S^1) \cong \ell^2(\mathbb{Z})$,

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[17] The existence of L^2 Fourier transform is rather non-trivial. While we do not provide a proof here, we should keep in mind its role in the story of Toeplitz operators, and in turn, we will use Toeplitz operators in the proof of Bott periodicity.

mapping the elements $\frac{1}{\sqrt{2\pi}}\psi_n \in L^2(S^1)$ to the canonical basis elements $\delta_n \in \ell^2(\mathbb{N})$. The Hardy space is then identified as $H^2(S^1) \cong \ell^2(\mathbb{N})$ under this isomorphism.

- The commutative C^*-algebra $C(S^1)$ is represented as bounded pointwise multiplication operators on $L^2(S^1)$,

  $$f \in C(S^1) \;\leftrightarrow\; M_f : \psi \mapsto f \cdot \psi, \;\; \psi \in L^2(S^1).$$

  Let us compress M_f to an operator acting on the Hardy subspace,

  $$T_f := P \circ M_f \circ \iota, \qquad f \in C(S^1), \tag{7.11}$$

  where $\iota : H^2(S^1) \hookrightarrow L^2(S^1)$ is the inclusion and $P = \iota^* : L^2(S^1) \to H^2(S^1)$ is the orthogonal projection. The operator T_f is called the *Toeplitz operator* with *symbol* f.

  We mention that

  $$T_f^* = (PM_f\iota)^* = \iota^* M_{\bar{f}} P^* = PM_{\bar{f}}\iota = T_{\bar{f}}. \tag{7.12}$$

  In other words, the "Toeplitzification map" $f \mapsto T_f$ is linear and $*$-preserving. However, it is not a homomorphism! (See (7.15) later.)

- Suppose $f \in C(S^1)$ is *invertible*, i.e., it is a continuous map $S^1 \to \mathrm{GL}(1,\mathbb{C})$, where $\mathrm{GL}(1,\mathbb{C}) = \mathbb{C}\backslash\{0\}$ is topologically the punctured plane. A basic result in algebraic topology is

  $$\pi_1(\mathrm{GL}(1,\mathbb{C})) \cong \mathbb{Z}, \tag{7.13}$$

  where $\pi_1(\cdot)$ denotes the *fundamental group* of homotopy classes of continuous loops $\ell : S^1 \to \mathrm{GL}(1,\mathbb{C})$.

  The bijection (7.13) is obtained as follows. First, it is straightforward to deform ℓ radially to obtain a loop $\ell : S^1 \to S^1$ taking values in the unit complex numbers. Regard ℓ as a path $[0, 2\pi] \to \mathrm{GL}(1,\mathbb{C})$ with matching endpoints,

  $$\ell(0) = \ell(2\pi).$$

  Let $\pi : \mathbb{R} \to S^1$, $\theta \mapsto e^{i\theta}$ be the universal covering map. Argue that ℓ is obtainable as $\ell = \pi \circ \tilde{\ell}$ for some lifted map[18] $\tilde{\ell} : [0, 2\pi] \to \mathbb{R}$, which need

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[18] The existence of such a lifted map constitutes the non-obvious part of the precise definition of winding number.

not be a loop. The condition

$$\pi(\tilde{\ell}(0)) = \ell(0) = \ell(2\pi) = \pi(\tilde{\ell}(2\pi))$$

requires

$$\tilde{\ell}(2\pi) = \tilde{\ell}(0) + 2\pi n$$

for some integer n, called the *winding number* of ℓ.

The typical loop with winding number $n \in \mathbb{Z}$ is

$$f_n : z = e^{i\theta} \mapsto e^{in\theta} = z^n.$$

- Since

  $$M_{f_1}\psi_m = \psi_{m+1}, \qquad m \in \mathbb{N},$$

  it is the Fourier transform of the right shift operator on $\ell^2(\mathbb{Z})$. So the Toeplitz operator T_{f_1} is the Fourier transform of the *unilateral right shift* S on $\ell^2(\mathbb{N})$,

  $$S : \delta_m \mapsto \delta_{m+1}, \qquad m \in \mathbb{N}.$$

  Clearly, S^* is the unilateral left shift operator, which annihilates δ_0. So S is not invertible. Indeed, S, or its Fourier transform T_{f_1}, is the simplest example of a Fredholm operator with nontrivial index, -1.

- **Theorem.** Let T_f be a Toeplitz operator with invertible symbol f. Then it is a Fredholm operator with

  $$\mathrm{Index}(T_f) = -\mathrm{Wind}(f). \tag{7.14}$$

  Proof:

  - Consider the set

    $$\Sigma = \bigl\{ f \in C(S^1) \,:\, \iota P M_f - M_f \iota P \in \mathcal{K}(L^2(S^1)) \bigr\}.$$

    This forms a (closed) C^*-subalgebra of $C(S^1)$. Observe that Σ includes the function $f_1 : z \mapsto z$. Consider what happens to each basis element $\psi_n : z \mapsto z^n$ of $L^2(S^1)$,

    $$(\iota P M_{f_1} - M_{f_1}\iota P)\psi_n = \begin{cases} 0, & n \geq 0, \\ \psi_0, & n = -1, \\ 0, & n < -1. \end{cases}$$

That is, $\iota P M_{f_1} - M_{f_1}\iota P$ is a rank-1 operator, thus compact.

By Stone–Weierstrass, the $*$-algebra generated by f_1 is dense in $C(S^1)$. Thus the C^*-algebra Σ is actually all of $C(S^1)$.

- Thus for all $f, f' \in C(S^1) = \Sigma$,

$$\begin{aligned} T_f T_{f'} &\equiv P M_f \iota P M_{f'} \iota \\ &= \underbrace{P\iota P}_{P} M_f M_{f'} \iota + \text{compact} \\ &= P M_{f\cdot f'} \iota + \text{compact} \\ &\equiv T_{f\cdot f'} + \text{compact}. \end{aligned} \tag{7.15}$$

In particular, for invertible symbol function $f \in C(S^1)$, we have

$$T_f T_{f^{-1}} = T_1 + \text{compact} = 1_{H^2(S^1)} + \text{compact},$$

i.e., T_f is invertible modulo compacts. By Atkinson's theorem (7.8), T_f is Fredholm.

- Note that if invertible functions $f, f' \in C(S^1)$ are homotopic, with

$$f_t : S^1 \to \mathrm{GL}(1, \mathbb{C}), \qquad t \in [0,1]$$

a homotopy, then $t \mapsto M_{f_t}$ is norm-continuous ($\|M_f\|_{\text{op}} = \|f\|_{\text{sup}}$), thus $t \mapsto T_{f_t}$ is a continuous path of Fredholm operators joining T_f and $T_{f'}$. By continuity of index (Eq. (7.10)), we deduce that the index of T_f only depends on the homotopy class of f, i.e., its winding number.

- Thus it suffices to compute the index of T_{f_n}, for the basic invertible functions $f_n : z \mapsto z^n$ representing the winding-n homotopy classes. Recall that $\psi_m : z \mapsto z^m, m \geq 0$ form a basis for $H^2(S^1)$.

($n \geq 0$ case.) We have

$$T_{f_n}\psi_m = P M_{f_n} \iota(\psi_m) = \psi_{m+n}, \qquad \forall m \geq 0,$$

and

$$T_{f_n}^* \psi_m = P M_{\overline{f_n}}(\psi_m) = P M_{f_{-n}}(\psi_m) = P\psi_{m-n} = \begin{cases} \psi_{m-n}, & m \geq n, \\ 0, & m < n. \end{cases}$$

Thus $\ker(T_{f_n}) = 0$ while $\ker(T^*_{f_n}) = \operatorname{span}\{\psi_m : m = 0, \ldots, n-1\}$, and we get

$$\operatorname{Index}(T_{f_n}) = \dim\ker(T_{f_n}) - \dim\ker(T^*_{f_n}) = -n.$$

($n < 0$ case.) A similar calculation again gives $\operatorname{Index}(T_{f_n}) = -n$. $\square$