16 Fibre bundles and structure groups

In Section 1.1.1, we emphasized that a vector $v \in V$ is a pairing of a basis/frame $\beta : \mathbb{K}^n \xrightarrow{\cong} V$ with numerical components $\mathbf{v} \in \mathbb{K}^n$, in many equivalent ways,

$$v = \beta(\mathbf{v}) = (\beta \circ g)(g^{-1}\mathbf{v}), \qquad \forall g \in \mathrm{GL}(n). \tag{16.1}$$

Therefore, we can recover V as a set of equivalence classes,

$$V = \Big(\text{Frames} \times \mathbb{K}^n\Big)/_{(\beta,\mathbf{v})\sim(\beta\cdot g, g^{-1}\mathbf{v})}. \tag{16.2}$$

Fixing a choice of β, we get representatives $(\beta, \mathbf{v})$ for the equivalence classes, then the right side of (16.2) inherits a vector space structure from $\mathbb{K}^n$. This vector space structure does not depend on the choice of β.

The construction (16.2) can be generalized to recover a vector bundle E from the principal frame bundle $\mathrm{Fr}(E)$. Let us formalize such constructions in a general language.

16.1 Associated fibre bundles

Let $\pi : P \to X$ be a principal G-bundle over X. Let ρ be a left G-action on a manifold F, meaning that

$$\begin{aligned} G \times F &\to F \\ (g, \xi) &\mapsto \rho(g) \cdot \xi \end{aligned} \tag{16.3}$$

is a smooth map satisfying

$$\rho(e) \cdot \xi = \xi, \qquad \rho(g_1 g_2) \cdot \xi = \rho(g_1) \cdot \rho(g_2) \cdot \xi, \qquad \forall\, g_1, g_2 \in G,\ \xi \in F.$$

For convenience, we will usually just write

$$g \cdot \xi \equiv \rho(g) \cdot \xi.$$

Given such a ρ, the set $P \times F$ is equipped with the right G-action,

$$(p, \xi) \cdot g := (p \cdot g, g^{-1} \cdot \xi),$$

and we may pass to the set of equivalence classes modulo this G-action,

$$P \times_\rho F := (P \times F)/_{(p,\xi)\sim(p\cdot g, g^{-1}\cdot\xi)}.$$

Notice that the map

$$\begin{aligned} \pi_\rho : P \times_\rho F &\to X \\ [p, \xi] &\mapsto \pi(p) \end{aligned}$$

is well-defined on the equivalence classes, since $\pi(p \cdot g) = \pi(p)$ for all $g \in G$. Give $P \times_\rho F$ the quotient topology, then $\pi_\rho : P \times_\rho F \to X$ is a continuous surjection (Exercise).

Exercise 16.1. For each $x \in X$, check that

$$\pi_\rho^{-1}(x) = \{[p, \xi] \,:\, \xi \in F\}$$

where p can be chosen to be any point in $P_x = \pi^{-1}(x)$.

Exercise 16.1 says that each fibre of $\pi_\rho : (P \times_\rho F) \to X$ is in *non-canonical* bijection with the *typical fibre* F. Below, we sketch how the fibres $\pi_\rho^{-1}(x)$ are smoothly assembled together.

Let $\Phi : \pi^{-1}(U) \to U \times G$ be a local trivialization of P. This is equivalently the local gauge

$$s_\Phi : U \to P, \qquad x \mapsto \Phi^{-1}(x, e).$$

"Attach" F-valued "components" to this local gauge,

$$\begin{aligned} \Phi^{(\rho)} : \pi_\rho^{-1}(U) &\to U \times F \\ [s_\Phi(x), \xi] &\mapsto (x, \xi), \end{aligned} \tag{16.4}$$

to get a continuous local trivialization of $\pi_\rho^{-1}(U)$.

Exercise 16.2. Let (U_α, Φ_α), (U_β, Φ_β) be two local trivializations of a principal G-bundle P, with smooth transition function $g_{\beta\alpha} : U_\beta \cap U_\alpha \to G$. Let $\Phi_\alpha^{(\rho)}, \Phi_\beta^{(\rho)}$ be the corresponding (continuous) local trivializations of $\pi_\rho : P \times_\rho F \to X$, as in (16.4). Verify that

$$\begin{aligned} \Phi_\beta^{(\rho)} \circ (\Phi_\alpha^{(\rho)})^{-1} : (U_\beta \cap U_\alpha) \times F &\to (U_\beta \cap U_\alpha) \times F \\ (x, \xi) &\mapsto (x, \rho(g_{\beta\alpha}(x))(\xi)). \end{aligned} \tag{16.5}$$

By smoothness of ρ in the sense of (16.3), the transition map (16.5) is smooth. So $P \times_\rho F$ is a manifold, with smooth projection map π_ρ, and admits local trivializations $\Phi_\alpha^{(\rho)}$ inherited from those of P.

Remark. A local trivialization $\Phi_\alpha^{(\rho)}$ tell us how to describe a local section of the vector bundle $P \times_\rho F$ in terms of F-valued functions. Sometimes, $\Phi_\alpha^{(\rho)}$ is also called a local gauge, and switching to a different local gauge $\Phi_\beta^{(\rho)}$ is called a local gauge transformation. Formula (16.5) describes how the F-valued descriptions are related under a local gauge transformation.

Definition 63. The structure $\pi_\rho : P \times_\rho F \to X$ constructed above is called the *fibre bundle associated with the principal G-bundle $\pi : P \to X$ and the action G-action ρ on the typical fibre F.*

More briefly, we just say that $P \times_\rho F$ is an *associated fibre bundle.* In the special case where ρ is a linear representation of G on a vector space $F = V$, then the associated fibre bundle is called an *associated vector bundle.*

Example 16.1. Recall that the frame bundle $\mathrm{Fr}(E)$ of a vector bundle E is a principal $\mathrm{GL}(n)$-bundle. Let ρ be the defining action of $\mathrm{GL}(n)$ on $\mathbb{K}^n$ by matrix multiplication. Then $\mathrm{Fr}(E) \times_\rho \mathbb{K}^n$ is an associated vector bundle with transition functions $\rho(g_{\beta\alpha}) = g_{\beta\alpha}$. Thus $\mathrm{Fr}(E) \times_\rho \mathbb{K}^n$ is isomorphic to E.

This isomorphism is canonical. Each element of the fibre $(\mathrm{Fr}(E) \times_\rho \mathbb{K}^n)_x$ is an equivalence class $[p, \mathbf{v}]$, representable by a frame $p \in \mathrm{Fr}(E)_x$ at x together with components $\mathbf{v} \in \mathbb{K}^n$ with respect to that frame. Note that $p : \mathbb{K}^n \xrightarrow{\cong} E_x$ is a basis. The equivalence class $[p, \mathbf{v}]$ is precisely the vector $p(\mathbf{v}) \in E_x$, and this vector is independent of the choice of basis p.

This is a suitable point to introduce the general notion of fibre bundles:

Definition 64. A *fibre bundle* with *typical fibre* F on which the *structure Lie group* G acts, is a smooth surjective map $\pi : E \to X$, such that every $x \in X$ lies in an open neighbourhood U with $E|_U := \pi^{-1}(U)$ being locally trivializable: there is a diffeomorphism

$$\Phi : E|_U \to U \times F \quad \text{such that} \quad \pi_U \circ \Phi = \pi \text{ on } E|_U.$$

Furthermore, for each pair $(U_\alpha, \Phi_\alpha), (U_\beta, \Phi_\beta)$ of local trivializations, we have

$$\Phi_\beta \circ \Phi_\alpha^{-1}(x, \xi) = (x, g_{\beta\alpha}(x) \cdot \xi), \qquad x \in U_\beta \cap U_\alpha, \xi \in F,$$

for smooth G-valued *transition functions* $g_{\beta\alpha} : U_{\alpha\beta} \to G$.

For a submanifold $U \subset X$, a smooth map $s : U \to E$ such that $\pi \circ s = \mathrm{id}_U$ is called a *(local) section* of E over U. The space of sections over U is denoted $\Gamma(E|_U)$. If $U = X$, then s is called a *global section* of E.

Example 16.2. When we gave up the notion of absolute space, we arrived at the notion of space-time, Definition 5, as a fibre bundle $\pi : \mathcal{N} \to \mathcal{T}$ over the timeline $\mathcal{T}$. Each fibre is identifiable with the model space $\mathcal{M}$ but not canonically so. Without further geometric structure on space-time, the structure group of this fibre bundle is $\mathrm{Diff}(M)$.

Example 16.3. As we saw in (15.7)–(15.8), a principal G-bundle is, in particular, a fibre bundle with typical fibre G, and structure Lie group G acting on G itself by left multiplication. Beware that left multiplication only makes sense locally, after identifying $P|_U$ with $U \times G$ — there is no global left G-action on the bundle P itself.

Example 16.4. A rank-n vector bundle is a fibre bundle, with structure group $\mathrm{GL}(n)$ acting on the typical fibre $\mathbb{K}^n$. In Section 16.4, we will put structure on vector bundles, and "reduce" its structure group to a subgroup of $\mathrm{GL}(n)$.

Remark. A fibre bundle may not have any global sections. But it always has local sections defined around any point $x \in X$, because of the local trivializability condition.

A vector bundle always has the zero section as a global section. But it may not admit any global *nowhere-vanishing* section.

16.2 Operations on vector bundles

On vector spaces E, F, we have algebraic operations,

- Duals E^*;
- Direct sums $E \oplus F$;
- (Symmetrized/antisymmetrized) tensor products $E \otimes F$;
- $\mathrm{Hom}(E, F) \cong F \otimes E^*$;
- Complexification/realification;
- Complex conjugate $\overline{E}$.

These algebraic operations generalize to operations on vector bundles E, F over X.

For example, we can take the direct sum $E \oplus F$ of a rank-m and a rank n-vector bundle. Use local trivializations $E|_U \cong U \times \mathbb{K}^m$ and $F|_U \cong U \times \mathbb{K}^n$

to construct $(E \oplus F)|_U \cong U \times (\mathbb{K}^m \oplus \mathbb{K}^m)$. The transition functions are valued in

$$\mathrm{GL}(m) \times \mathrm{GL}(n) \subset \mathrm{GL}(m+n),$$

so $E \oplus F$ is a rank-$(m+n)$ vector bundle.

We can construct the contangent bundle T^*X as an associated bundle to the the tangent frame bundle $\mathrm{Fr}(TX)$ — take $\mathrm{GL}(n)$ to act on $\mathbb{R}^n$ in the *contragredient representation* $g \mapsto (g^{-1})^{\mathrm{t}}$.

Similarly for tensor product bundles. We will return to this when we discuss differential forms.

16.3 Vector bundle metrics and orientation

If E is a real vector bundle, then a *Euclidean bundle metric* is a section

$$\langle \cdot, \cdot \rangle_E \in \Gamma(E^* \otimes E^*)$$

which restricts to an inner product $\langle \cdot, \cdot \rangle_x$ on each E_x, $x \in X$. Then E is called a *Euclidean vector bundle.* The pointwise inner products of two sections give a real-valued function,

$$\langle \psi, \tilde{\psi} \rangle_E \in C^\infty(X)$$
$$\langle \psi, \tilde{\psi} \rangle_E(x) := \langle \psi(x), \tilde{\psi}(x) \rangle_x.$$

For example, a *Riemannian metric* on X is a Euclidean bundle metric on TX.

Similarly, if E is a complex vector bundle, then a *Hermitian bundle metric* is a section

$$\langle \cdot, \cdot \rangle_E \in \Gamma(\overline{E}^* \otimes E^*)$$

which restricts at each $x \in X$ to a Hermitian inner product on E_x. Then E is called a *Hermitian vector bundle.*

Two bases for a real vector space V are said be *equivalently oriented* if the change-of-basis matrix has positive determinant. An *orientation* of V is a choice of equivalence class of bases. The set $\mathcal{O}_V$ of orientations is a $\mathbb{Z}_2$-torsor.

Let E be a real vector bundle. Recall that a local trivialization $\Phi_\alpha : U_\alpha \to U_\alpha \times \mathbb{R}^n$ determines a local frame,

$$x \mapsto (s_1(x), \ldots, s_n(x)), \qquad s_i(x) = \Phi_\alpha^{-1}(x, e_i),\ x \in U_\alpha.$$

So we have a basis, thus an orientation, for each E_x, $x \in U_\alpha$. For another local trivialization $\Phi_\beta : U_\alpha \to U_\alpha \times \mathbb{R}^n$, we another another orientation on E_x for $x \in U_\beta \cap U_\alpha$. The two orientations agree iff the transition function has

$$\operatorname{sgn}(\det g_{\beta\alpha}(x)) = 1.$$

A bundle atlas is *oriented* if all the transition functions have positive determinant. In this case, we obtain unambiguous orientations $\mathcal{O}_{E_x}$ on E_x for all $x \in X$, by taking the basis furnished by any Φ_α in the atlas. We say that E is *orientable* if it admits an oriented bundle atlas. An *orientation* on E is a choice of maximal oriented bundle atlas (which then determines a global orientation assignment $x \mapsto \mathcal{O}_{E_x}$).

16.3.1 Restricted frame bundles

If E is a Euclidean vector bundle, we can restrict attention to local trivializations which preserve the metric on the fibres (i.e., the E_x are identified with $\mathbb{R}^n$ as inner product spaces). The transition functions will then become $\mathrm{O}(n)$-valued. The local frames determined by such local trivializations will be local *orthonormal frames.* There is an *orthonormal frame bundle* $\mathrm{Fr}^{\mathrm{O}}(E)$ inside the full frame bundle $\mathrm{Fr}(E)$. In the same way as $\mathrm{Fr}(E)$, the subbundle $\mathrm{Fr}^{\mathrm{O}}(E)$ has a principal $\mathrm{O}(n)$-bundle structure.

We recover E as a Euclidean vector bundle, by taking the associated vector bundle construction with $\mathbb{R}^n$ given the standard inner product, and $\mathrm{O}(n)$ acting in the defining representation on $\mathbb{R}^n$.

Similarly, if E is an oriented vector bundle, its *oriented frame bundle* is a principal $\mathrm{GL}(n, \mathbb{R})^+$-bundle. If E is an oriented Euclidean vector bundle, then its *oriented orthonormal frame bundle* is a principal $\mathrm{SO}(n)$-bundle.

If E is a Hermitian vector bundle, its *orthonormal frame bundle* is a principal $\mathrm{U}(n)$-bundle, denoted $\mathrm{Fr}^{\mathrm{U}}(E)$.

Remark. For $E = TX$, the frame bundles are often just called the frame bundles of the base (oriented, Riemannian) manifold X, with the vector bundle TX being implicit.

16.4 Gauge transformations of vector bundles with reduced structure group

The *endomorphism bundle* of a vector bundle E is $\mathrm{End}(E) := E \otimes E^*$. The terminology arises because $T \in \Gamma(\mathrm{End}(E))$ acts on $v \in \Gamma(E)$ in the obvious pointwise manner,

$$T : \Gamma(E) \to \Gamma(E), \qquad (T \cdot v)(x) = T(x)(v(x)),$$

and $T \cdot (fv) = fT(v)$, for all $f \in C^\infty(X)$. Elements of $\Gamma(\mathrm{End}(E))$ are called *bundle endomorphisms*, and they can be linearly combined, and composed. There is an identity bundle endomorphism, and the invertible bundle endomorphisms form a group, denoted $\mathrm{GL}(E)$.

If E has some extra structure (such as a metric on the fibres), and one restricts to local trivializations which preserve this structure, then the transition functions will be valued in a *reduced* structure group $G \subset \mathrm{GL}(n)$. For example, $G = \mathrm{U}(n), \mathrm{O}(n), \mathrm{SO}(n)$ etc. The bundle E is then regarded as a fibre/vector bundle with reduced structure group G.

We can also restrict attention to those bundle endomorphisms $T \in \mathrm{GL}(E)$ for which each $T(x)$ preserves the extra structure on E_x. Such a bundle endomorphism T is called a *gauge transformation of* E as a vector bundle with structure group G. These restricted bundle endomorphisms form a subgroup, denoted $\mathcal{G}(E)$.

Let $F \in \mathcal{G}(P)$ be a gauge transformation of a principal G-bundle P. We know that F is represented as right-multiplication by an equivariant map $\sigma_F : P \to G$. On any associated vector bundle $E = P \times_\rho V$, there is a corresponding gauge transformation, as follows. For $[p, \xi] \in E$, define

$$T_F[p, \xi] := [p \cdot \sigma_F(p), \xi] \equiv [p, \rho(\sigma_F(p)) \cdot \xi], \tag{16.6}$$

One checks that (Exercise)

- The map T_F in Eq. (16.6) defines a bundle endomorphism of E as a vector bundle with structure group $\rho(G)$.
- If ρ is a faithful representation (i.e. injective), then $F \mapsto T_F$ is a group isomorphism $\mathcal{G}(P) \to \mathcal{G}(E)$.

16.5 Reduction of structure group

The frame bundles $\mathrm{Fr}^{\mathrm{O}}(E) \subset \mathrm{Fr}(E)$ etc., are examples of a general notion of reduction of structure group.

Definition 65. Let $\phi : H \to G$ be a Lie group homomorphism, $\pi : P \to X$ be a principal H-bundle, and $\pi' : P' \to X$ be a principal G-bundle. A map $F : P \to P'$ is called a *ϕ-reduction of P'* if

- $\pi' \circ F = \pi$,
- $F(p \cdot h) = F(p) \cdot \phi(h), \qquad h \in H, p \in P.$

In particular, if ϕ is the embedding of a Lie subgroup $H \subset G$, we simply call P a *H-reduction of P'*.

Remark. The map F restricts to a map $P_x \to P'_x$ of fibres. Picking $p \in P_x$ and $F(p) \in P'_x$ gives identifications $P_x \cong H$ and $P'_x \cong G$. Then $F|_{P_x}$ is identified with the homomorphism $\phi : H \to G$.

Let P be a principal G-bundle, and consider an associated vector bundle $E = P \times_\rho V$. Suppose V is equipped with a G-invariant inner product,

$$\langle \xi, \zeta \rangle_V = \langle \rho(g)\xi, \rho(g)\zeta \rangle_V, \qquad \xi, \zeta \in V,\ g \in G.$$

In other words, the representation ρ of G on the typical fibre V is unitary/orthogonal. Then the vector bundle $E = P \times_\rho V$ acquires the bundle metric

$$\langle [p, \xi], [p, \zeta] \rangle_E := \langle \xi, \zeta \rangle_V. \tag{16.7}$$

With this extra metric structure, the structure group of E is reduced from $\mathrm{GL}(V)$ to $\rho(G) \subset \mathrm{U}(V)$ or $\rho(G) \subset \mathrm{O}(V)$.

16.6 Scalar fields?

What if the structure group of a fibre bundle E is completely reduced to the trivial group? This means that E has a bundle atlas with trivial transition functions. So local trivializations $\Phi_\alpha : E|_{U_\alpha} \to U_\alpha \times F$ and $\Phi_\beta : E|_{U_\beta} \to U_\beta \times F$ can be combined into a single local trivialization $U_{\alpha\beta} : E|_{U_\alpha \cup U_\beta} \to (U_\alpha \cup U_\beta) \times F$, because they are the same map on the overlap. Continuing this process over an open cover, we get a single global trivialization $E \xrightarrow{\cong} X \times F$. So sections of

E become canonically identified with F-valued functions, $\Gamma(E) = C^\infty(X; F)$. Why is it useful to think of F-valued functions as sections of a fibre bundle with trivial transition functions?

In general, the classical differential geometry of X is encoded in its frame bundle, with structure group G being some subset of $\mathrm{GL}(n)$ depending on what extra geometric structures X has. The tangent vector bundle is obtained as a bundle associated to the defining representation of G. But other representations of G could also be considered, e.g., trivial representation, tensor product representation, contragredient representation, etc. Quantum mechanically, we would even consider even projective unitary representations of G, the famous examples being the half-integer "spin" representations of $G = \mathrm{SO}(3)$. The sections of the resulting associated vector bundles are called *fields* in physics, and they come in various types — vector, n-forms, spinor, scalar — according to the representation of G.

The particular case of the trivial representation on V corresponds to V-valued *scalar fields* (sometimes called "spin-0" fields in the case of $G = \mathrm{SO}(n)$). Notice that the adjective "scalar" is used, despite the "vector" values. The defining characteristic of a "scalar" field is that local frame rotations do not affect how the field is described as a V-valued function. This is completely different from the situation of, e.g., tangent vector fields.

In the context of quantum mechanics, a projective representation of G lifts to a genuine representation of a central extension $\tilde{G}$,

$$1 \to \mathrm{U}(1) \to \tilde{G} \to G \to 1,$$

in which the central subgroup $\mathrm{U}(1)$ is represented as scalar multiplication on a complex inner product space V. So $\tilde{G}$ is not trivially represented, even when G is trivially represented (or even when G is just the trivial group). As such, we do not actually have scalar quantum mechanical wavefunctions, but only sections of associated bundles to $\tilde{G}$. Furthermore, this $\mathrm{U}(1)$ has physical meaning in relation to the electromagnetic force.

In the modern theory of elementary particles and strong/weak nuclear forces, there are other "internal symmetry groups" G which have nothing to do with the geometry of X. Then there is a principal G-bundle of abstract frames for the "internal local degrees of freedom", which is completely distinct from the classical frame bundle of X. This gives another motivation to study general fibre bundle geometry.