# 18 Geometry of Lie group actions

## 18.1 Lie algebra of left-invariant vector fields

**Definition 73.** A vector field $v$ on a Lie group $G$ is *left-invariant* if

$$(L_g)_* v = v, \qquad \forall g \in G.$$

Using Exercise 17.2, we have

$$(L_g)_*[v, w] = [(L_g)_* v, (L_g)_* w] = [v, w]$$

for left-invariant vector fields $v, w$ on $G$. Thus the commutator preserves the property of left-invariance.

**Definition 74.** The *Lie algebra* of a Lie group $G$, denoted $\mathfrak{g}$, is the vector space of left-invariant vector fields on $G$, equipped with the *Lie bracket* $[\cdot, \cdot]$ of commutator of vector fields.

**Proposition 18.1.** *The Lie algebra $\mathfrak{g}$ of a Lie group $G$ is linearly isomorphic to $T_eG$ via evaluation at the identity element $e \in G$.*

*Proof.* Linearity and injectivity of the evaluation-at-$e$ map is straightforward. For surjectivity, pick $\xi \in T_eG$. A left-invariant $v$ having $v_e = \xi$ must have

$$v_g = ((L_g)_* v)_g = (dL_g)_e(v_e) = (dL_g)_e(\xi) \in T_gG, \qquad g \in G. \tag{18.1}$$

It remains to check that the above $v : g \mapsto v_g$ is a smooth vector field. By Exercise 14.1, we need to check that when $v$ is applied to arbitrary $f \in C^\infty(G)$, the resulting function

$$\begin{aligned} v(f) : G &\to \mathbb{R} \\ g &\mapsto v_g(f) \end{aligned}$$

is smooth. Let $\gamma : (-\delta, \delta) \to G$ be a curve in $G$ with $\gamma(0) = e$ and $\dot{\gamma}(0) = \xi$. Then

$$v_g(f) = ((dL_g)_e(\xi))\,(f) = \xi\,(f \circ L_g) = \frac{d(f \circ L_g \circ \gamma)}{dt}\Big|_{t=0}. \tag{18.2}$$

The right side is the partial derivative $\frac{\partial}{\partial t}|_{t=0}$ of the smooth function

$$(-\delta, \delta) \times G \to \mathbb{R}, \qquad (t, g) \mapsto f \circ L_g \circ \gamma(s),$$

so it is smoothly dependent on $g \in G$. ☐

Thus $\mathfrak{g}$ has finite linear dimension equal to the manifold dimension of $G$. The Lie bracket of $\mathfrak{g}$ is transferred to $T_eG$ via the canonical identification $T_eG \cong \mathfrak{g}$ of Prop. 18.1.

**Corollary 18.2.** *A Lie group is* parallelizable*, i.e., its tangent bundle is trivializable.*

*Proof.* Pick any basis $\{\xi_1, \ldots, \xi_n\}$ for $T_eG$. By Prop. 18.1, Eq. (18.1), we obtain the left-invariant vector fields

$$v^{(i)} : g \mapsto (dL_g)(\xi_i), \qquad i = 1, \ldots, \dim G.$$

Since the $dL_g$ are linear isomorphisms, the $v^{(i)}$ define a global frame for $TG$, thus $TG$ is trivializable (Exercise 14.3). □

*Example* 18.1. Write $\mathfrak{gl}(n, \mathbb{R})$ for the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$. Regard $\mathrm{GL}(n, \mathbb{R})$ an open submanifold of the vector space $\mathrm{M}_n(\mathbb{R}) = \mathbb{R}^{n^2}$. Then

$$\mathfrak{gl}(n, \mathbb{R}) = T_e\mathrm{GL}(n) \cong T_e\mathrm{M}_n(\mathbb{R}) = \mathrm{M}_n(\mathbb{R}).$$

The matrix commutator provides a Lie bracket operation[24] for $\mathrm{M}_n(\mathbb{R})$. This Lie bracket coincides with the one inherited from the vector field Lie bracket on $\mathfrak{gl}(n, \mathbb{R})$, via the identification $\mathfrak{gl}(n, \mathbb{R}) \cong \mathrm{M}_n(\mathbb{R})$ (Exercise). Similarly, the Lie bracket on $\mathfrak{gl}(n, \mathbb{C}) = \mathrm{M}_n(\mathbb{C})$ coincides with the matrix commutator.

**Definition 75.** A *Lie group homomorphism* $f : G \to H$ is a smooth group homomorphism, while a *Lie algebra homomorphism* $\mathfrak{g} \to \mathfrak{h}$ is a Lie bracket-preserving linear map.

*Exercise* 18.1. Let $f : G \to H$ be a Lie group homomorphism. Regard

$$f_* := (df)_e : T_eG \to T_eH$$

as a linear map $\mathfrak{g} \to \mathfrak{h}$ through the identification of Prop. 18.1. Show that $f_*$ is actually a Lie algebra homomorphism.

*Exercise* 18.2. Show that if $G$ is an abelian Lie group, then its Lie algebra $\mathfrak{g}$ has trivial Lie bracket, $[u, v] = 0$ for all $u, v \in \mathfrak{g}$.

[24] A Lie bracket operation on a vector space $V$ is an antisymmetric bilinear map $V \times V \to V$ satisfying the Jacobi identity,

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0, \qquad u, v, w, \in V.$$

## 18.2 Matrix Lie group and algebra examples

Often, an abstract Lie group $G$ is isomorphic to some *matrix Lie group*, i.e., a closed subgroup of $\mathrm{GL}(n)$ such as $\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n)$. In this case, the Lie algebra $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R}) \cong \mathrm{M}_n(\mathbb{R})$ or $\mathfrak{gl}(n, \mathbb{C}) \cong \mathrm{M}_n(\mathbb{C})$. Then the Lie bracket on $\mathfrak{g}$ can be identified with the matrix commutator (recall Example 18.1).

*Exercise* 18.3. Show that

- $\mathfrak{o}(n) = \mathfrak{so}(n) \subset \mathrm{M}_n(\mathbb{R})$ is the space of antisymmetric matrices.
- $\mathfrak{u}(n) \subset \mathrm{M}_n(\mathbb{C})$ is the space of antihermitian matrices.
- $\mathfrak{su}(n) \subset \mathrm{M}_n(\mathbb{C})$ is the space of traceless antihermitian matrices.

Non-isomorphic Lie groups can have isomorphic Lie algebras. For example, the Lie algebras of $\mathrm{GL}(n, \mathbb{R})^+$ and $\mathrm{GL}(n, \mathbb{R})$ coincide; similarly, $\mathfrak{o}(n) = \mathfrak{so}(n)$. A more interesting example is $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. Take

$$\begin{aligned} \mathfrak{su}(2) &= \mathrm{span}_{\mathbb{R}} \left\{ \frac{1}{2i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{2i}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \\ \mathfrak{so}(3) &= \mathrm{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}. \end{aligned} \tag{18.3}$$

A direct calculation shows that the linear map exchanging the above bases is a Lie algebra isomorphism (Exercise).

## 18.3 Adjoint representation

Conjugation of $G$ by any element $g \in G$,

$$C_g : G \to G, \qquad g' \mapsto g g' g^{-1},$$

is a group automorphism of $G$. In fact, each $C_g$ is a diffeomorphism, thus a *Lie group automorphism.* Therefore, $C_g$ induces a Lie algebra automorphism (Exercise 18.1),

$$\mathrm{Ad}_g := (C_g)_* = (dC_g)_e : \mathfrak{g} \to \mathfrak{g}.$$

In particular, $\mathrm{Ad}_g \in \mathrm{GL}(\mathfrak{g})$ for each $g \in G$. By the chain rule,

$$\mathrm{Ad}_{g_1} \circ \mathrm{Ad}_{g_2} = \mathrm{Ad}_{g_1 g_2}.$$

So the map

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}), \qquad g \mapsto \mathrm{Ad}_g$$

gives a representation of $G$ as linear operators on the vector space $\mathfrak{g}$, called the *adjoint representation.*

*Exercise* 18.4. Let $v \in \mathfrak{g}$ be a left-invariant vector field on $G$. Show that $(R_g)_* v$ is also left-invariant, and that

$$(R_g)_* v = \mathrm{Ad}_{g^{-1}}(v), \qquad \forall\, g \in G.$$

*Example* 18.2. Consider $\mathrm{GL}(n) \subset \mathrm{M}_n$ with Lie algebra $\mathfrak{gl}(n) = \mathrm{M}_n$. With matrix entries as coordinates for $G$, each conjugation $C_g$ is described as a linear map of these coordinates. So

$$\begin{aligned} \mathrm{Ad}_g \equiv (dC_g)_e : \mathfrak{gl}(n) &\to \mathfrak{gl}(n) \\ A &\mapsto gAg^{-1}, \qquad g \in \mathrm{GL}(n). \end{aligned} \tag{18.4}$$

Likewise, for a matrix Lie group $G \subset \mathrm{GL}(n)$, the adjoint representation of $G$ on $\mathfrak{g} \subset \mathrm{M}_n$ is obtained by restricting the conjugations, (18.4).

## 18.4 Exponential map $\mathfrak{g} \to G$

**Proposition 18.3.** *Left-invariant vector fields on Lie groups are complete.*

*Proof.* Let $v$ be a left-invariant vector field on $G$. There is an integral curve $\gamma^{(e)} : (-\delta, \delta) \to G$ for some $\delta > 0$. By left-invariance, for any $g \in G$, the translated curve $L_g \circ \gamma^{(e)}$ is the integral curve starting at $g$, and it is also defined for $t \in (-\delta, \delta)$. By the uniform time Lemma 17.2, $v$ is complete. ☐

By Prop. 18.3, given $v \in \mathfrak{g}$, there is an integral curve $\gamma_v : \mathbb{R} \to G$ starting at $\gamma_v(0) = e$. Pick some other point $\gamma_v(s), s \in \mathbb{R}$ on this curve. The integral curve starting at $\gamma_v(s)$ is obtained in either of the following two ways:

- $t \mapsto (L_{\gamma_v(s)} \circ \gamma_v)(t) = \gamma_v(s) \cdot \gamma_v(t)$; or
- $t \mapsto \gamma_v(s + t)$.

These two formulae coincide,

$$\gamma_v(s) \cdot \gamma_v(t) = \gamma(s + t), \qquad s, t \in \mathbb{R}.$$

Thus $\gamma_v : \mathbb{R} \to G$ is actually a *Lie group homomorphism.*

**Definition 76.** A *one-parameter subgroup of a Lie group $G$* is a curve $\gamma : \mathbb{R} \to G$ which is also a Lie group homomorphism.

**Theorem 18.4.** *One-parameter subgroups of a Lie group $G$ are in bijection with left-invariant vector fields on $G$.*

*Proof.* Starting from a left-invariant $v \in \mathfrak{g}$, we had just seen that its integral curve starting at the identity is a one-parameter subgroup.

In reverse, a one-parameter subgroup $\gamma : \mathbb{R} \to G$ determines an initial velocity $\dot{\gamma}(0) \in T_eG = \mathfrak{g}$, therefore a left-invariant vector field,

$$g \mapsto (dL_g)_e(\dot{\gamma}(0)). \tag{18.5}$$

We need to check that $\gamma$ is indeed an integral curve of (18.5). First,

$$\dot{\gamma}(s) = (d\gamma)_s \left( \frac{d}{dt}\Big|_{t=s} \right) \in T_{\gamma(s)}G, \qquad s \in \mathbb{R}.$$

The homomorphism property $\gamma(s+t) = \gamma(s)\gamma(t)$ is equivalently rewritten as $\gamma \circ L_s = L_{\gamma(s)} \circ \gamma$, and this implies that

$$d\gamma \circ dL_s = dL_{\gamma(s)} \circ d\gamma, \qquad s \in \mathbb{R}.$$

Then

$$\begin{aligned} \dot{\gamma}(s) = (d\gamma)_s \left( \frac{d}{dt}\Big|_{t=s} \right) &= d\gamma_s \left( (dL_s)_0 \left( \frac{d}{dt}\Big|_{t=0} \right) \right) \\ &= (dL_{\gamma(s)})_e \left( d\gamma_0 \left( \frac{d}{dt}\Big|_{t=0} \right) \right) = (dL_{\gamma(s)})_e(\dot{\gamma}(0)), \end{aligned} \tag{18.6}$$

which is the required integral curve condition. □

**Definition 77.** For a Lie group $G$ with Lie algebra $\mathfrak{g}$, the *exponential map* is defined as

$$\exp : \mathfrak{g} \to G, \qquad v \mapsto \gamma_v(1),$$

where $\gamma_v : \mathbb{R} \to G$ is 1-parameter subgroup with $\dot{\gamma}_v(0) = v$.

For $t \neq 0$, $\gamma_{tv}$ is just a reparametrization of $\gamma_v$ with parameter $s$ rescaled to $st$, thus

$$\exp(tv) = \gamma_{tv}(1) = \gamma_v(t).$$

One says that $v \in \mathfrak{g}$ *generates* the one-parameter subgroup

$$t \mapsto \gamma_v(t) = \exp(tv). \tag{18.7}$$

*Example* 18.3. The *matrix exponential* is the map

$$\mathrm{M}_n(\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R}), \qquad A \mapsto e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

(Convergent in operator norm, say.) For $A \in \mathrm{M}_n(\mathbb{R})$, consider the map

$$\gamma_A : \mathbb{R} \to \mathrm{GL}(n,\mathbb{R}), \qquad t \mapsto e^{tA}.$$

Using matrix entries $X^{ij}$ as coordinates on $\mathrm{GL}(n,\mathbb{R})$, we see that $\gamma_A$ is a smooth map which satisfies

$$\begin{aligned} \gamma_A(0) &= 1_n, \\ \gamma_A(s+t) &= e^{(s+t)A} = e^{sA}e^{tA} = \gamma_A(s)\gamma_A(t), \qquad s,t \in \mathbb{R}, \\ \dot{\gamma}_A(0) &= \sum_{i,j=1}^{n} A^{ij} \frac{\partial}{\partial X^{ij}}\Big|_{1_n} \in T_{1_n}\mathrm{GL}(n,\mathbb{R}). \end{aligned}$$

So, with the usual identification $T_{1_n}\mathrm{GL}(n,\mathbb{R}) = \mathfrak{gl}(n,\mathbb{R})$, the curve $\gamma_A$ is the one-parameter subgroup of $\mathrm{GL}(n,\mathbb{R})$ corresponding to $A \in \mathfrak{gl}(n,\mathbb{R})$. Thus the Lie-theoretic exponential map (Definition 77) is realized by the ordinary matrix exponential. Similarly for the complex case.

*Example* 18.4. Consider the non-compact matrix Lie group $(\mathbb{R}_{>0}, \times) = \mathrm{GL}(1,\mathbb{R})$. The Lie algebra is $\mathfrak{gl}(1,\mathbb{R}) = \mathbb{R}$, spanned by the tangent vector $\partial_x|_1$. The exponential map is

$$\mathfrak{gl}(1,\mathbb{R}) = \mathbb{R} \ni t \mapsto \exp(t\partial_x|_1) = e^t \in \mathrm{GL}(1,\mathbb{R}).$$

Compare with the compact matrix Lie group $\mathrm{U}(1) \subset \mathrm{GL}(1,\mathbb{C})$, which has Lie algebra $\mathfrak{u}(1)$ being the real 1-dimensional subspace $i\mathbb{R} \subset \mathrm{M}_1(\mathbb{C}) = \mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$, on which the commutator is trivial. The exponential map is

$$\mathfrak{u}(1) = i\mathbb{R} \ni it \mapsto \exp(it) = e^{it} \in \mathrm{U}(1).$$

Clearly $\mathfrak{gl}(1,\mathbb{R}) \cong \mathfrak{u}(1)$ as abstract Lie algebras. However, $\mathrm{GL}(1,\mathbb{R})$ and $\mathrm{U}(1)$ are non-isomorphic Lie groups. This emphasizes that the exponential map depends on which Lie group is being referred to.

*Exercise* 18.5. Let $f : G \to H$ be a Lie group homomorphism, and $f_* : \mathfrak{g} \to \mathfrak{h}$ be the induced Lie group homomorphism. Show that the exponential map is natural, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{f_*} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{f} & H \end{array} .$$

**Proposition 18.5.** *Let $f : G \to H$ be a Lie group homomorphism. The adjoint representations of $G$ and $H$ are compatible with $f$ in the sense that*

$$f_* \circ \mathrm{Ad}_g = \mathrm{Ad}_{f(g)} \circ f_*, \qquad g \in G.$$

*Proof.* Let $g \in G$ and $v \in \mathfrak{g} = T_eG$. From Exercise 18.5, we have

$$f(g \exp(tv)\, g^{-1}) = f(g) f(\exp(tv)) f(g)^{-1} = f(g) \exp(t f_*(v)) f(g)^{-1},$$

equivalently,

$$f \circ C_g \circ \exp(tv) = C_{f(g)} \circ \exp(t f_*(v)).$$

Differentiate at $t = 0$ to obtain

$$f_* \circ \mathrm{Ad}_g(v) = \mathrm{Ad}_{f(g)} \circ f_*(v).$$

□

## 18.5 Maurer–Cartan form

Generalizing the concept of $\mathbb{R}$-valued 1-forms, we may consider the space $\Omega^1(X, V)$ of 1-forms on $X$ with values in a fixed vector space $V$.

**Definition 78.** The Maurer–Cartan form on a Lie group $G$ is the $\mathfrak{g}$-valued 1-form $\Theta \in \Omega^1(G, \mathfrak{g})$, defined by

$$\Theta_g(v_g) := (L_{g^{-1}})_*(v_g), \qquad v_g \in T_gG. \tag{18.8}$$

The idea behind $\Theta$ is to "push" each tangent space $T_gG$ back to $T_eG = \mathfrak{g}$, using the canonical left group action $L$. This reverses the global frame construction in Corollary 18.2.

*Exercise* 18.6. Show that the Maurer–Cartan form $\Theta$ on a Lie group $G$ is

- Fixes the left-invariant vector fields,

$$\Theta(v) = v, \qquad \forall\, v \in \mathfrak{g}; \tag{18.9}$$

- Right-equivariant,

$$(R_g)^*\Theta = \mathrm{Ad}_{g^{-1}} \circ \Theta. \tag{18.10}$$

For $G = \mathrm{GL}(n)$, the Maurer–Cartan form is usually written as

$$\Theta = g^{-1}dg.$$

This notation means that at $g' \in \mathrm{GL}(n,\mathbb{R})$, we have the matrix $(g')^{-1}$ multiplied into the matrix of coordinate 1-forms at $g'$,

$$(g^{-1}dg)_{g'} = (g')^{-1} \cdot \begin{pmatrix} (dX^{11})_{g'} & \cdots & (dX^{1n})_{g'} \\ \vdots & \ddots & \vdots \\ (dX^{n1})_{g'} & \cdots & (dX^{nn})_{g'} \end{pmatrix}, \tag{18.11}$$

thus $g^{-1}dg$ is a $\mathrm{M}_n(\mathbb{R}) = \mathfrak{gl}(n,\mathbb{R})$-valued 1-form. One checks that $g^{-1}dg$ is indeed the Maurer–Cartan form of $\mathrm{GL}(n)$ according to Definition 78.

For $\mathrm{GL}(n,\mathbb{C})$, the coordinate $X^{ij}$ is replaced by a complex coordinate $Z^{ij} = X^{ij} + iY^{ij}$. The Maurer–Cartan form has the same formula, with $dX^{ij}$ replaced by $dZ^{ij}$.

For a matrix Lie group $G \subset \mathrm{GL}(n)$, the Maurer–Cartan form is still given by the same formula, but the 1-forms $dX^{ij}$ are restricted to $TG$.

*Exercise* 18.7. Work out the Maurer–Cartan form explicitly for the matrix Lie groups $\mathrm{U}(1)$ and $\mathrm{SU}(2)$.

## 18.6 Fundamental vector fields on spaces with $G$-action

Let $G$ act smoothly on another manifold $P$ on the right (by diffeomorphisms). We denote this action by the smooth map

$$\sigma : P \times G \to P, \qquad (p, g) \mapsto p \cdot g.$$

Fix $v \in \mathfrak{g}$, and restrict to the action of the 1-parameter subgroup $\gamma_v = \exp(\,\cdot\, v) : \mathbb{R} \to G$,

$$\begin{aligned} \sigma : P \times \mathbb{R} &\to P \\ (p, t) &\mapsto p \cdot \exp(tv). \end{aligned}$$

Starting at $p \in P$, we take the curve $t \mapsto p \cdot \exp(tv)$. The velocity vector at $t = 0$ is denoted[25]

$$v^\sharp_p := \frac{d}{dt}\Big|_{t=0} (p \cdot \exp(tv)) \in T_pP. \tag{18.12}$$

**Definition 79.** Let $G$ act on a manifold $P$. Given $v \in \mathfrak{g}$, the vector field $v^\sharp$ defined by Eq. (18.12) is called the *fundamental vector field* on $P$ associated to $v$.

Note that the integral curves of $v^\sharp$ are obtained simply by acting with $\exp(tv)$ on the right.

*Example* 18.5. Let $P = G$, with $G$ acting on itself by right multiplication. If $v \in \mathfrak{g}$ is a left-invariant vector field on $G$, then at any $g \in G$,

$$v^\sharp_g = \frac{d}{dt}\Big|_{t=0} (g \cdot \exp(tv)) = (L_g)_* \left( \frac{d}{dt}\Big|_{t=0} (\exp(tv)) \right) = (L_g)_*(v_e) = v_g.$$

So in this case, $v^\sharp = v$ itself.

*Example* 18.6. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, whose points are labelled by unit row vectors $(n_x, n_y, n_z)$. The group SO(3) acts on unit row vectors by right matrix-multiplication. Consider the $\mathfrak{so}(3)$ element $J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. A calculation of the matrix exponential shows that $J_z$ generates the 1-parameter subgroup

$$t \mapsto \exp(tJ_z) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This subgroup effects

$$(n_x, n_y, n_z) \cdot \exp(tJ_z) = (n_x \cos t + n_y \sin t, -n_x \sin t + n_y \cos t, n_z)$$

[25] The sharp notation is standard, and should not be confused with the musical isomorphism introduced earlier.

which is a clockwise rotation by angle $t$ about the $z$-axis. The fundamental vector field is

$$\begin{aligned}(J_z)^\sharp_{(n_x,n_y,n_z)} &= \frac{d}{dt}\Big|_{t=0}(n_x\cos t + n_y \sin t, -n_x \sin t + n_y \cos t, n_z)\\ &= (n_y, -n_x, 0) \in T_{(n_x,n_y,n_z)}S^2.\end{aligned}$$

*Exercise* 18.8. Show that the map $\mathfrak{g} \to \mathfrak{X}(P)$ taking $v \mapsto v^\sharp$ is $G$-equivariant, in the following sense,

$$(R_g)_*(v^\sharp) = (\mathrm{Ad}_{g^{-1}}(v))^\sharp, \qquad \forall g \in G$$

### 18.6.1 Derivative of $G$-action

For later use, we record the following calculation:

**Proposition 18.6.** *Let $\sigma : P \times G \to P$ denote the action of a Lie group $G$ on a manifold $P$. Then*

$$d\sigma_{(p,g)}(v,\xi) = d(R_g)_p(v) + (\Theta(\xi))^\sharp_{p\cdot g}, \qquad (v,\xi) \in T_pP \oplus T_gG, \qquad (18.13)$$

*where $\Theta$ is the Maurer–Cartan form on $G$.*

*Proof.* First consider $(v, 0) \in T_pP \oplus T_gG$. We may represent $v$ by a curve $\gamma$ in $P$ with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Then $(v, 0)$ is represented by the curve $t \mapsto (\gamma(t), g)$ in $P \times G$. By Prop. 13.1,

$$d\sigma_{(p,g)}(v,0) = \frac{d(\sigma(\gamma(t),g))}{dt}\Big|_{t=0} = \frac{d(R_g\circ\gamma(t))}{dt}\Big|_{t=0} = d(R_g)_p(\dot\gamma(0)) = d(R_g)_p(v).$$

Next, consider $(0, \xi) \in T_pP \oplus T_gG$. Via the Maurer–Cartan form, $\xi$ is identified with $\Theta(\xi) \in \mathfrak{g} = T_eG$, and the latter is the velocity vector of $t \mapsto \exp(t\Theta(\xi))$. So $\xi \in T_gG$ is the velocity vector of $t \mapsto g\exp(t\Theta(\xi))$. Applying $d\sigma$ to $(0, \xi) \in T_pP \oplus T_gG$ gives the element of $T_{p\cdot g}G$ which is the velocity vector of $t \mapsto p \cdot g \exp(t\Theta(\xi))$, i.e., the fundamental vector field $(\Theta(\xi))^\sharp$ at $p \cdot g$.

□

# 19 Connections on principal bundles and their curvature

## 19.1 Canonical vertical bundle of a principal bundle

Let $\pi : P \to X$ be a principal $G$-bundle. Each $p \in P$ lies in some fibre $P_{\pi(p)}$, which is a $G$-torsor (diffeomorphic to $G$). So we may define the *vertical tangent subspace* at $p$,

$$V_p := T_p P_{\pi(p)} \subset T_p P,$$

which has $\dim V_p = \dim G$. A vertical tangent vector is geometrically represented by a curve lying within a single fibre. After applying $\pi$, such a "vertical curve" becomes a constant curve on the base $X$, thus

$$V_p \subset \ker d\pi_p.$$

Because $\pi$ is a submersion, $d\pi_p$ is surjective, so

$$\dim\, \ker d\pi_p = \dim P - \dim X = \dim G.$$

Thus

$$V_p = \ker d\pi_p. \tag{19.1}$$

The *vertical bundle* of $P$ is the following subset of $TP$,

$$VP := \ker d\pi = \bigsqcup_{p\in P} V_p \subset \bigsqcup_{p\in P} T_p P = TP,$$

with projection map $VP \to P$ inherited from $TP \to P$.

For $\xi \in \mathfrak{g}$, the fundamental vector field $\xi^\sharp$ at $p \in P$ is represented by the "vertical" curve $t \mapsto p \cdot \exp(t\xi)$. Indeed, we have linear isomorphisms

$$\begin{aligned} \mathfrak{g} = T_e G &\xrightarrow{\cong} T_p P_{\pi(p)} = V_p \\ \xi &\mapsto \xi^\sharp_p = \frac{d}{dt}\Big|_{t=0} p \cdot \exp(t\xi). \end{aligned} \tag{19.2}$$

Eq. (19.2) is a restriction of $d\sigma$, where $\sigma$ is the right-action map (Eq. (18.13)). So the map

$$\begin{aligned} P \times \mathfrak{g} &\to VP \subset TP \\ (p, \xi) &\mapsto \xi^\sharp_p \end{aligned} \tag{19.3}$$

is a smooth bijection, which can be used to give a smooth global frame for $VP$. This implies that $VP$ is a vector subbundle[26] of $TP$. In fact, (19.3) provides a canonical vector bundle isomorphism

$$\sharp^{-1} : VP \xrightarrow{\cong} P \times \mathfrak{g}, \tag{19.4}$$

thus a global trivialization of $VP$ once a basis for $\mathfrak{g} \cong \mathbb{R}^{\dim G}$ is picked. This generalizes $TG \cong G \times \mathfrak{g}$ (Corollary 18.2).

## 19.2 Connection as horizontal bundle

To specify "horizontal" directions in a principal bundle complementary to the canonical vertical ones, extra geometric data must be provided.

**Definition 80.** A *connection* on a principal $G$-bundle $P$ is a choice of $G$-invariant *horizontal subbundle* $HP \to P$ of $TP \to P$, i.e.,

$$\begin{aligned} HP \oplus VP &= TP, \\ (R_g)_* H_p &= H_{pg}, \qquad \forall p \in P,\ g \in G. \end{aligned}$$

The second condition is based on the gauge principle — gauge-covariant constructions should treat all points within a fibre of $P$ on an equal footing.

Once a connection is specified, the derivative map $d\pi : TP \to TX$ has its

$$d\pi_p : T_pP = H_p \oplus V_p \to T_{\pi(p)}X, \qquad p \in P$$

restricting to isomorphisms $d\pi_p|_{H_p} : H_p \xrightarrow{\cong} T_{\pi(p)}X$, due to (19.1).

**Definition 81.** Let $\pi : P \to X$ be a principal $G$-bundle with a connection. Given a tangent vector field $u \in \mathfrak{X}(X)$ on the base, its *horizontal lift* is the unique vector field $\tilde{u}^{\mathrm{H}} \in \Gamma(HP) \subset \Gamma(TP)$ such that the following commutes:

$$\begin{array}{ccc} P & \xrightarrow{\tilde{u}^{\mathrm{H}}} & HP \\ \pi \downarrow & & \downarrow d\pi|_{HP} \\ X & \xrightarrow{u} & TX \end{array}.$$

[26] A subset $F$ of a rank-$n$ vector bundle $\pi : E \to X$ is a vector subbundle of rank $k \leq n$, if every $x \in X$ is contained in a local trivialization $\Phi : E|_U \xrightarrow{\cong} U \times \mathbb{K}^n$ such that $\Phi(E|_U \cap F) = U \times \mathbb{K}^k$. Then $F$ is an embedded submanifold in $E$, and $\pi|_F : F \to X$ is itself a rank-$k$ vector bundle.

*Exercise* 19.1. Show that for any $u \in \mathfrak{X}(X)$, its horizontal lift $\tilde{u}^{\mathrm{H}}$ is $G$-invariant, i.e., $(R_g)_*\tilde{u}^{\mathrm{H}} = \tilde{u}^{\mathrm{H}}$ for all $g \in G$.

*Exercise* 19.2. Let $v \in \Gamma(TP)$ be $G$-invariant, and $\xi^\sharp$ be a fundamental vector field on $P$ (thus $\xi^\sharp$ is vertical). Show that $[v, \xi^\sharp] = 0$.

## 19.3 Connection 1-form

Let $P$ be equipped with a connection. Then we have splittings

$$T_pP \ni v_p = v_p^{\mathrm{H}} + v_p^{\mathrm{P}} \in H_p \oplus V_p.$$

Likewise, any vector field $v \in \Gamma(TP)$ splits into a horizontal and a vertical horizontal part,

$$v = v^{\mathrm{H}} + v^{\mathrm{V}}, \qquad v^{\mathrm{H}} \in \Gamma(HP),\ v^{\mathrm{V}} \in \Gamma(VP).$$

Using the canonical isomorphism (19.4), we have the following $\mathfrak{g}$-valued 1-form $\omega$ on $P$,

$$\begin{aligned} \omega : \mathfrak{X}(P) \longrightarrow\ & \Gamma(VP) \stackrel{\sharp^{-1}\circ}{\longrightarrow} \Gamma(P \times \mathfrak{g}) = C^\infty(P; \mathfrak{g}) \\ v \mapsto\ & \quad v^{\mathrm{V}} \quad \mapsto \quad \sharp^{-1} \circ v^{\mathrm{V}}. \end{aligned} \tag{19.5}$$

The 1-form $\omega$ has the following properties:

- $\omega$ annihilates the horizontal parts of vector fields;
- $\omega$ takes a fundamental vector field $\xi^\sharp$ back to $\xi$;

$$\omega(\xi^\sharp) = \xi, \qquad \forall\, \xi \in \mathfrak{g}. \tag{19.6}$$

- $\omega$ is $G$-equivariant,

$$R_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega, \qquad \forall g \in G. \tag{19.7}$$

The first property and the second property, Eq. (19.6), follow from the defining formula (19.5) for $\omega$. The third equivariance property, Eq. (19.7), may be deduced from Exercise 18.4 and the first property (Exercise).

Our second definition of a connection is:

**Definition 82.** A connection on a principal $G$ bundle $P$ is a $\mathfrak{g}$-valued 1-form on $P$ which satisfies (19.6)–(19.7).

Start from a connection $\omega$ in the sense of Definition 82. Define the horizontal subspaces of $T_pP$ to be

$$H_p = \ker(\omega_p), \qquad p \in P,$$

so they are annihilated by $\omega$. These $H_p$ assemble into a vector subbundle $HP \subset TP$, complementary[27] to $VP$ by Condition (19.6). As an Exercise, condition (19.7) may be shown to imply $G$-invariance of $HP$. Thus we recover the notion of a connection in the sense of Definition 80.

*Example* 19.1. Consider the underlying manifold of $G$ as a principal $G$-bundle over a point. In Example 18.5, we saw that $\xi^\sharp = \xi$, $\xi \in \mathfrak{g}$. So Exercise 18.6 shows that the Maurer–Cartan form $\Theta$ is a connection on $G$.

*Example* 19.2. A trivial principal bundle $X \times G$ possesses a *trivial connection* $\omega_{\text{triv}} = \pi_G^*\Theta$, where $\Theta$ is the Maurer–Cartan form on $G$. Thus $\omega_{\text{triv}}$ is just the Maurer–Cartan form on each fibre $\{x\} \times G$, while the horizontal subspaces are $H_{(x,g)} = T_xX \oplus 0$.

If $P$ is trivializable, we may use a trivialization $\Phi : P \stackrel{\cong}{\to} X \times G$ to pull $\omega_{\text{triv}}$ back to a connection on $P$. *There is no intrinsic "trivial connection" on $P$ without specifying a trivialization.*

*Example* 19.3. Let $P = S^1 \times \mathrm{U}(1)$. Let $d\theta, d\varphi$ be the respective coordinate 1-forms on $S^1$ and $\mathrm{U}(1)$ (Example 17.4). For $ik \in \mathfrak{u}(1) = i\mathbb{R}$, the corresponding fundamental vector field on $P$ is $k\frac{\partial}{\partial\varphi}$. The Maurer–Cartan form on $\mathrm{U}(1)$ is $id\varphi$. Consider the $\mathfrak{u}(1)$-valued 1-form $\omega^{(k)} := ik\,d\theta + id\varphi$. Then $\omega^{(k)}$ is a connection on $P$. The horizontal subspaces are spanned by $\partial_\theta - k\partial_\varphi$.

*Exercise* 19.3. Let $\omega, \omega'$ be two connections on a principal $G$ bundle $P$. Show that $\omega - \omega'$ is a *horizontal* 1-form in the sense that it annihilates all vertical vectors,

$$(\omega - \omega')_p(v_p) = 0, \qquad \forall\, v_p \in V_p,\ p \in P.$$

[27] Technically, one may put a Riemannian metric on $P$, making each $H_p$ the orthogonal complement to $V_p$; then $HP$ is readily seen to be a vector subbundle. See Theorem 5.2.2 of Hamilton for details.

## 19.4 Local description of connection: gauge potentials

Connections are somewhat abstract, so for concrete computations, one describes them locally, with respect to some local trivialization/gauge.

**Definition 83.** Let $\omega$ be a connection on a principal $G$-bundle $P$. Let $s : U \to P$ be a local section/gauge over an open subset $U \subset X$. Then $s^*\omega$ is a $\mathfrak{g}$-valued 1-form over $U$, called a *local gauge potential*.

By definition,

$$s^*\omega_x(u_x) = \omega_{s(x)}(ds_x(u_x)) = \sharp^{-1}\big(ds_x(v_x)^{\mathrm{V}}\big), \qquad v_x \in T_xX.$$

Intuitively, $s^*\omega$ measures the failure of $s$ to be "horizontal" — $ds$ lands within $HP$, then $s^*\omega \equiv 0$.

*Example* 19.4. On the trivial principal bundle $S^1 \times \mathrm{U}(1)$, consider the connection 1-form $\omega = ikd\theta + id\varphi$ of Example 19.3. The trivial section $s_0 : e^{i\theta} \mapsto (e^{i\theta}, 1)$ is not horizontal with respect to $\omega$. Indeed, the gauge potential is $s_0^*\omega = ikd\theta$. This means that when the base coordinate is increased infinitesimally by $d\theta$, the section "increases" in the fibre direction by an infinitesimal amount $ik \in \mathfrak{u}(1)$.

**Theorem 19.1.** *Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection on a principal $G$-bundle $P$. Let $s_\alpha, s_\beta$ be local sections of $P$ defined over open subsets $U_\alpha, U_\beta$ respectively. On the overlap $U_\alpha \cap U_\beta$, the respective local gauge potentials are related by*

$$s_\beta^*\omega = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} \circ s_\alpha^*\omega + g_{\alpha\beta}^*\Theta, \tag{19.8}$$

*where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is the local gauge transformation implementing $s_\beta = s_\alpha \cdot g_{\alpha\beta}$, and $\Theta$ is the Maurer–Cartan form on $G$.*

*Proof.* With $\sigma : P \times G \to P$ the action map, we have $s_\beta = \sigma \circ (s_\alpha, g_{\alpha\beta})$. Its derivative is

$$d(s_\beta)_x = d\sigma_{(s_\alpha(x), g_{\alpha\beta}(x))} \circ (d(s_\alpha)_x, d(g_{\alpha\beta})_x), \qquad x \in U_\alpha \cap U_\beta. \tag{19.9}$$

The formula for $d\sigma$, Eq. (18.13), is recalled below,

$$d\sigma_{(p,g)}(v, \xi) = d(R_g)_p(v) + (\Theta(\xi))^\sharp_{p\cdot g}.$$

Substitute this into Eq. (19.9) and evaluate on some $\eta \in T_xX$,

$$\begin{aligned} d(s_\beta)_x(\eta) &= d(R_{g_{\alpha\beta}(x)})_{s_\alpha(x)}(d(s_\alpha)_x(\eta)) \ + \ (\Theta(d(g_{\alpha\beta})_x(\eta)))^\sharp_{s_\beta(x)} \\ &= d(R_{g_{\alpha\beta}(x)})_{s_\alpha(x)}(d(s_\alpha)_x(\eta)) \ + \ ((g^*_{\alpha\beta}\Theta)_x(\eta))^\sharp_{s_\beta(x)}. \end{aligned} \tag{19.10}$$

For convenience, we drop the reference to $x$. Apply $\omega$ to the first term of Eq. (19.10),

$$\begin{aligned} \omega\left(d(R_{g_{\alpha\beta}})_{s_\alpha}(d(s_\alpha)(\eta))\right) &= (R^*_{g_{\alpha\beta}}\omega)(d(s_\alpha)(\eta)) \\ &= \mathrm{Ad}_{g^{-1}_{\alpha\beta}}\big(\omega(d(s_\alpha)(\eta))\big) \qquad \text{(Eq. (19.7))} \\ &= \mathrm{Ad}_{g^{-1}_{\alpha\beta}}(s^*_\alpha\omega(\eta)). \end{aligned}$$

Apply $\omega$ to the second term of Eq. (19.10),

$$\omega((g^*_{\alpha\beta}\Theta)(\eta))^\sharp_{s_\beta} \overset{\text{Eq. (19.6)}}{=} (g^*_{\alpha\beta}\Theta)(\eta)$$

In total,

$$\underbrace{\omega(d(s_\beta)(\eta))}_{=s^*_\beta\omega(\eta)} = (\mathrm{Ad}_{g^{-1}_{\alpha\beta}} \circ s^*_\alpha\omega + g^*_{\alpha\beta}\Theta)(\eta),$$

which is the desired transformation law, Eq. (19.8). □

*Exercise* 19.4. Eq. (19.8) can be viewed as the action of a *local* gauge transformation $g_{\alpha\beta}$ on $s^*_\alpha\omega$ to obtain $s^*_\beta\omega$ (on the overlapping region $U_\alpha \cap U_\beta$). Let $s_\gamma : U_\gamma \to P$ be a third local section. Show that on $U_\alpha \cap U_\beta \cap U_\gamma$, we can obtain $s^*_\gamma\omega$ from $s^*_\alpha\omega$, by first applying $g_{\alpha\beta}$, then applying $g_{\beta\gamma}$.

*Remark.* When $G$ is abelian, the transformation law, Eq. (19.8), simplifies to

$$s^*_\beta\omega = s^*_\alpha\omega + g^*_{\alpha\beta}\Theta. \tag{19.11}$$

## 19.5 Local versus global picture of connections

A gauge potential is a *local, gauge-dependent* description of a connection. Nevertheless, a $\mathfrak{g}$-valued 1-form $\mathcal{A} \in \Omega^1(U, \mathfrak{g})$ is enough to specify a connection on the trivial principal bundle $U \times G$: set

$$\omega_{(x,g)}(v_x, \xi^\sharp_g) = \mathrm{Ad}_{g^{-1}} \circ \mathcal{A}(v_x) + \xi, \qquad v_x \in T_xU,\ \xi \in T_eG = \mathfrak{g}, \tag{19.12}$$

and check (Exercise) that it is a connection whose gauge potential in the trivial gauge $x \mapsto (x, e)$ is precisely $\mathcal{A}$.

In general, there is no global trivialization available. Suppose we have an open cover $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ of $X$, and transition functions $g_{\alpha\beta}$ satisfying the cocycle condition (15.9). So we have "patching data" for a principal $G$-bundle $P$; there are local gauges $s_\alpha$ identifying $P|_{U_\alpha} \cong U_\alpha \times G$, with $g_{\alpha\beta}$ being the transition functions. Now suppose we have a collection of $\mathfrak{g}$-valued local 1-forms $\mathcal{A}_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$, such that

$$\mathcal{A}_\beta = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} \circ \mathcal{A}_\alpha + g_{\alpha\beta}^*(\Theta) \quad \text{over } U_\alpha \cap U_\beta, \qquad \alpha, \beta \in \mathcal{I}.$$

Then it may be shown that there exists a *unique* connection $\omega$ on $P$ such that $\mathcal{A}_\alpha = s_\alpha^*\omega$ for each $\alpha \in \mathcal{I}$. In physics, a connection is often specified (and understood/defined) as such a coherent collection of $\mathcal{A}_\alpha$, and is usually called a *gauge field*.

Finally, we recall from Exercise 19.3 that any connection $\omega$ can be obtained from a reference connection $\omega_0$ by adding a horizontal $G$-equivariant $\mathfrak{g}$-valued 1-form $\eta$ on $P$. In turn, the space of such $\eta$ is canonically identified with the 1-forms on $X$ valued in the *adjoint bundle* $\mathrm{Ad}\, P = P \times_{\mathrm{Ad}} \mathfrak{g}$ (Exercise). So the space of connections is an affine space over an infinite-dimensional vector space.

## 19.6 Curvature of connections

The exterior derivative of a $k$-form generalizes to a map from $V$-valued $k$-forms to $V$-valued $(k+1)$-forms. Concretely, we pick a basis for $V$ and expand a $V$-valued $k$-form into $\mathbb{R}$-valued $k$-forms, then apply the usual exterior derivative on each component.

**Definition 84.** Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form on a principal $G$-bundle $P$. Its *curvature* is the $\mathfrak{g}$-valued 2-form

$$\Omega(u, v) = d\omega(u^{\mathrm{H}}, v^{\mathrm{H}}), \qquad u, v \in \mathfrak{X}(P). \tag{19.13}$$

Recall Eq. (17.11),

$$d\omega(u, v) = u(\omega(v)) - v(\omega(u)) - \omega([u, v]).$$

Since $\omega$ annihilates horizontal vector fields,

$$\Omega(u, v) = d\omega(u^{\mathrm{H}}, v^{\mathrm{H}}) = -\omega([u^{\mathrm{H}}, v^{\mathrm{H}}]). \tag{19.14}$$

**Intuitively:** In $P$, imagine moving an infinitesimally small distance horizontally along $u^{\mathrm{H}}$, then horizontally along $v^{\mathrm{H}}$. Now do this in the opposite order. If the horizontal distribution defined by $\omega$ is "curved", there will be a small vertical mismatch at the endpoint (an element of $V_p \cong \mathfrak{g}$). The curvature form measures this vertical mismatch.

Recall that a connection 1-form is $G$-equivariant (Eq. (19.7)). Similarly:

**Proposition 19.2.** *The curvature* $\Omega$ *of a connection* $\omega$ *is* $G$*-equivariant,*

$$R_g^*\Omega = \mathrm{Ad}_{g^{-1}} \circ \Omega, \qquad \forall\, g \in G.$$

*Proof.* Because $(R_g)_*$ respects the horizontal-vertical splitting of tangent spaces, we have

$$(R_g)_* v^{\mathrm{H}} = ((R_g)_* v)^{\mathrm{H}}, \qquad v \in \mathfrak{X}(P).$$

So for any $u, v \in \mathfrak{X}(P)$, we have

$$\begin{aligned}
(R_g^*\Omega)(u,v) &= \Omega((R_g)_*u, (R_g)_*v) \\
&= d\omega(((R_g)_*u)^{\mathrm{H}}, ((R_g)_*v)^{\mathrm{H}}) \\
&= d\omega((R_g)_*u^{\mathrm{H}}, (R_g)_*v^{\mathrm{H}}) \\
&= (R_g^*(d\omega))(u^{\mathrm{H}}, v^{\mathrm{H}}) \\
&= d(R_g^*\omega)(u^{\mathrm{H}}, v^{\mathrm{H}}) \\
&= d(\mathrm{Ad}_{g^{-1}} \circ \omega)(u^{\mathrm{H}}, v^{\mathrm{H}}) \\
&= \mathrm{Ad}_{g^{-1}} \circ (d\omega)(u^{\mathrm{H}}, v^{\mathrm{H}}) = \mathrm{Ad}_{g^{-1}} \circ \Omega(u,v).
\end{aligned}$$

□

**Definition 85.** The *wedge product* of $\eta, \omega \in \Omega^1(X, \mathfrak{g})$ is the $\mathfrak{g}$-valued 2-form defined by

$$[\eta, \omega](u,v)(x) = [\eta(u)(x), \omega(v)(x)] - [\eta(v)(x), \omega(u)(x)], \qquad u, v \in \mathfrak{X}(X),$$

where $[\cdot,\cdot]$ on the right side denotes the Lie bracket in $\mathfrak{g}$.

This is like an ordinary wedge product, but instead of pointwise multiplying $\eta(u)$ and $\omega(v)$, we take the pointwise Lie bracket. In the same way, we can take wedge products of $\mathfrak{g}$-valued $k$-forms and $l$-forms.

**Theorem 19.3** (Cartan structure equation)**.** *The curvature of a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ can be expressed as*

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \tag{19.15}$$

*Proof.* Exercise. It is helpful to split the calculation of $\Omega(u, v)$ into three cases: (i) $u, v$ both horizontal, (ii) $u, v$ both vertical, (iii) $u$ vertical and $v$ horizontal. □

*Remark.* In the particular case where $P = G$, we have no horizontal directions at all, and $\omega = \Theta$ is the Maurer–Cartan form. So $d\Theta + \frac{1}{2}[\Theta, \Theta] = 0$.

### 19.6.1 Local description: field strength

**Definition 86.** Let $\Omega$ be the curvature of a connection $\omega$ on a principal $G$-bundle $P$. Let $s : U \to P$ be a local section of $P$. Then $s^*\Omega \in \Omega^2(U, \mathfrak{g})$ is called the *local field strength* of $\omega$.

Because pullback is compatible with wedge product and the exterior derivative, it is straightforward to verify the local form of the Cartan structure equation,

$$s^*\Omega = d(s^*\omega) + \frac{1}{2}[s^*\omega, s^*\omega]. \tag{19.16}$$

Local field strengths are gauge-dependent, so let us work out how they transform when the local gauge is changed.

**Theorem 19.4.** *Let $\Omega$ be the curvature of a connection $\omega$ on a principal $G$-bundle. Let $s_\alpha : U_\alpha \to P$ and $s_\beta : U_\beta \to P$ be two local gauges. On $U_\alpha \cap U_\beta$, the local field strengths are related by*

$$s_\beta^*\Omega = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} \circ s_\alpha^*\Omega, \tag{19.17}$$

*where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is the local gauge transformation, $s_\beta = s_\alpha \cdot g_{\alpha\beta}$.*

*Proof.* We need to compute

$$s_\beta^*\Omega(u, v) = \Omega(ds_\beta(u), ds_\beta(v)), \qquad u, v \in \mathfrak{X}(U).$$

While proving the transformation law for the local gauge potentials, Theorem 19.1, we derived Eq. (19.10),

$$ds_\beta(u) = dR_{g_{\alpha\beta}} \circ ds_\alpha(u) + (g_{\alpha\beta}^*\Theta)(u))^\sharp, \qquad u \in \mathfrak{X}(U),$$

where we suppressed the basepoints for convenience. The second term is vertical, so it does not contribute to $\Omega(\cdot,\cdot)$. Thus for any $u, v \in \mathfrak{X}(X)$,

$$\begin{aligned}(s_\beta^*\Omega)(u,v) &= \Omega(dR_{g_{\alpha\beta}} \circ ds_\alpha(u), dR_{g_{\alpha\beta}} \circ ds_\alpha(v)) \\ &= R^*_{g_{\alpha\beta}}\Omega(ds_\alpha(u), ds_\alpha(v)) \\ &= \mathrm{Ad}_{g^{-1}_{\alpha\beta}} \circ \Omega(ds_\alpha(u), ds_\alpha(v)) && \text{(Prop. 19.2)} \\ &= \mathrm{Ad}_{g^{-1}_{\alpha\beta}} \circ s_\alpha^*\Omega(u,v).\end{aligned}$$

□

## 19.7 Globally gauge-transformed connections and curvatures

Let $F : P_1 \to P_2$ be a morphism of principal $G$-bundles, and let $\omega \in \Omega^1(P_2, \mathfrak{g})$ be a connection on $P_2$. Then

$$\begin{aligned}R_g^*(F^*\omega) = (F \circ R_g)^*\omega &= (R_g \circ F)^*\omega && \text{(morphism)} \\ &= F^*(R_g^*\omega) \\ &= F^*(\mathrm{Ad}_{g^{-1}} \circ \omega) && \text{(Eq. (19.7))} \\ &= \mathrm{Ad}_{g^{-1}} \circ (F^*\omega).\end{aligned}$$

Furthermore, if $v^\sharp$ is a fundamental vector field on $P_1$, then

$$\begin{aligned}(dF)_p(v^\sharp_p) &= (dF)_p\left(\frac{d(p \cdot \exp(tv))}{dt}\bigg|_{t=0}\right) \\ &= \frac{d(F(R_{\exp(tv)}(p)))}{dt}\bigg|_{t=0} \\ &= \frac{d(R_{\exp(tv)}(F(p)))}{dt}\bigg|_{t=0} && \text{(morphism)} \\ &= v^\sharp_{F(p)},\end{aligned}$$

so

$$(F^*\omega)_p(v^\sharp_p) = \omega_{F(p)}((dF)_p(v^\sharp_p)) = \omega_{F(p)}(v^\sharp_{F(p)}) = v, \qquad \forall\, p \in P_1.$$

We learn that $F^*\omega \in \Omega^1(P_1, \mathfrak{g})$ satisfies the defining properties (19.7) and (19.6) for a connection on $P_1$.

*In particular, (global) gauge transformations of $P$ act on connections on $P$ by pullback.*

**Proposition 19.5.** *Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection on a principal $G$-bundle $P$, and let $F \in \mathcal{G}(P)$ be a gauge transformation, implemented by right multiplication by an equivariant map $\sigma_F \in \mathrm{Map}(P, G)^G$ as in Eq.* (15.14). *Then*

$$F^*\omega = \mathrm{Ad}_{\sigma_F^{-1}} \circ \omega + \sigma_F^*\Theta. \tag{19.18}$$

*Proof.* The calculation follows that in Theorem 19.1, and is omitted. □

**Theorem 19.6.** *Let $\omega$ be a connection on a principal $G$-bundle $P$. Let $\sigma_F \in \mathrm{Map}(P, G)^G = \mathcal{G}(P)$ be a gauge transformation. Then*

$$\Omega^{F^*\omega} = F^*\Omega^\omega = \mathrm{Ad}_{\sigma_F^{-1}} \circ \Omega^\omega.$$

*where $\Omega^\omega$ and $\Omega^{F^*\omega}$ are the respective curvatures of $\omega$ and $F^*\omega$.*

*Proof.* For the first equality, we use the Cartan structure equation, and commutativity of pullback with wedge product,

$$\begin{aligned} F^*\Omega^\omega &= F^*\left(d\omega + \frac{1}{2}[\omega, \omega]\right) \\ &= d(F^*\omega) + \frac{1}{2}[F^*\omega, F^*\omega] = \Omega^{F^*\omega}. \end{aligned}$$

For the second equality, let $u, v \in \mathfrak{X}(P)$. In the expression

$$F^*\Omega^\omega(u, v) = \Omega^\omega\left(F_*u, F_*v\right), \tag{19.19}$$

we may discard the vertical parts of the arguments on the right. To understand $F_* = dF$, we note that $F : P \to P$ may be written as the composition

$$p \mapsto (p, \sigma_F(p)) \overset{\sigma}{\mapsto} p \cdot \sigma_F(p),$$

where $\sigma : P \times G \to P$ is the group action map. We computed $d\sigma$ in Eq. (18.13),

$$d\sigma_{(p,g)}(v_p, \xi_g) = (dR_g)_p(v_p) + \ldots, \qquad v_p \in T_pP,\ \xi_g \in T_gG,$$

where we suppressed the remaining vertical term. Substitute this into Eq. (19.19),

$$\begin{aligned} F^*\Omega^\omega(u, v) &= \Omega^\omega\left(dR_{\sigma_F(p)}(u), dR_{\sigma_F(p}(v)\right) \\ &= R^*_{\sigma_F}\Omega^\omega(u, v) \\ &= \mathrm{Ad}_{\sigma_F^{-1}} \circ \Omega^\omega\left(u, v\right), \qquad\qquad u, v \in \mathfrak{X}(P), \end{aligned}$$

where the last equality used the $G$-equivariance of $\Omega^\omega$. □

Theorem 19.6 is the global version of Theorem 19.4.

### 19.7.1 Abelian $G$

Theorem 19.6 says that when $G$ is abelian, the curvature $\Omega \in \Omega^2(P, \mathfrak{g})$ is invariant under gauge transformations. In particular, the local field strengths $s^*\Omega$ are independent of the choice of local gauge/section $s$. (This can also be seen from Eq. (19.17).) This means that the $s^*\Omega$ can be regarded as the restriction of a single globally-defined 2-form $\mathcal{F} \in \Omega^2(X, \mathfrak{g})$.

Furthermore, this $\mathcal{F}$ is a *closed* 2-form: Computing in any local gauge $s$, $\mathcal{F} = s^*\Omega = s^*(d\omega) = d(s^*\omega)$ holds, so

$$d\mathcal{F} = d(d(s^*\omega)) = 0.$$

Nevertheless, it is important to remember that there may not exist a *global* gauge $s$ such that $\mathcal{F} = d(s^*\omega)$ holds. The closed 2-form $\mathcal{F}$ is said to be *locally exact* but not necessarily *globally exact*, and the failure is measured by the class of $\mathcal{F}$ in the *second de Rham cohomology group* of $X$ (Optional). The latter group is a topological invariant of $X$.

## 19.8 Physics notation

The physicists' notation for the local gauge potentials and local field strengths are $\mathcal{A}$ and $\mathcal{F}$, where

$$\mathcal{A} = s^*\omega, \qquad \mathcal{F} = s^*\Omega,$$

and reference to a local gauge $s$ is left implicit. One takes the local structure equation, Eq. (19.16),

$$\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}], \tag{19.20}$$

as the *definition* of the (local) curvature 2-form of $\mathcal{A}$.

Furthermore, one typically works in a coordinate chart $(U, \varphi)$, and expands $\mathcal{A}$ in terms of coordinate 1-forms. So $\mathcal{A}$ (more precisely, $(\varphi^{-1})^*\mathcal{A}$) is written as

$$\mathcal{A} = \sum_{i=1}^{d} \mathcal{A}_i dx^i,$$

where each component $\mathcal{A}_i$ is a smooth $\mathfrak{g}$-valued function on $\varphi(U)$. Similarly, $\mathcal{F} = \sum_{1 \le i < j \le d} \mathcal{F}_{ij} dx^i \wedge dx^j$, where (Exercise)

$$\mathcal{F}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i + [\mathcal{A}_i, \mathcal{A}_j].$$

Next, $G$ is usually a matrix Lie group, $G \subset \mathrm{GL}(n)$, so $\Theta = g^{-1}dg$ (see Eq. (18.11)). Then the second term in the gauge transformation law, Eq. (19.8), is

$$g_{\alpha\beta}^{*}\Theta = g_{\alpha\beta}^{-1} \cdot \begin{pmatrix} d(g_{\alpha\beta}^{11}) & \cdots & d(g_{\alpha\beta}^{1n}) \\ \vdots & \ddots & \vdots \\ d(g_{\alpha\beta}^{n1}) & \cdots & d(g_{\alpha\beta}^{nn}) \end{pmatrix},$$

with $g_{\alpha\beta}^{ij} = X^{ij} \circ g_{\alpha\beta}$ being the $ij$-th matrix entry of the transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n)$. It is common practice to simply write $g = g_{\alpha\beta}$, and condense the above expression into "$g^{-1}dg$". So the transformation rules under a local gauge transformation $g$ are:

$$\begin{aligned} \mathcal{A} &\rightsquigarrow \mathcal{A}' = g^{-1}\mathcal{A}g + g^{-1}dg \\ \mathcal{F} &\rightsquigarrow \mathcal{F}' = g^{-1}\mathcal{F}g. \end{aligned} \tag{19.21}$$

*Remark.* Sometimes, $g$ is replaced by $g^{-1}$ to get a left action of local gauge transformations, and there is a factor of $i$ relating physicists' "matrix Lie algebras" and the mathematicians' Lie algebras. The matrix identity $dg = -g(dg^{-1})g$ may be used to write $-dg^{-1}g$ instead of $g^{-1}dg$.