7 Unbounded operators on Hilbert space

7.1 Basic definitions for unbounded operators

With differential operators in mind, we make the following generalization:

Definition 20. An *operator* on a Banach space V is a linear map $T : \mathrm{Dom}(T) \to V$, whose *domain* $\mathrm{Dom}(T)$ is some subspace of V. Its *graph* is the subset

$$\Gamma_T = \{(\psi, T\psi) \,:\, \psi \in \mathrm{Dom}(T)\} \subset V \oplus V.$$

T is said to be

- *Closed*, if $\Gamma_T = \overline{\Gamma_T}$ is a closed set;
- *Closable*, if $\overline{\Gamma_T}$ is the graph of some operator $\overline{T}$; in this case, $\overline{T}$ is called the *closure* of T.

Definition 21. Let T, T' be operators on a Banach space. By default,

$$\mathrm{Dom}(TT') = \{v \in \mathrm{Dom}(T') \,:\, T'v \in \mathrm{Dom}(T)\}$$
$$\mathrm{Dom}(T + T') = \mathrm{Dom}(T) \cap \mathrm{Dom}(T').$$

If

$$\mathrm{Dom}(T) \subset \mathrm{Dom}(T'), \qquad T'|_{\mathrm{Dom}(T)} = T,$$

we write $T \subset T'$.

Exercise 7.1. Let $S \in \mathcal{B}(V)$ be bounded, and $T : \mathrm{Dom}(T) \to V$ be an operator.

- Suppose $\mathrm{Ran}(S) \subset \mathrm{Dom}(T)$. Show that $TS \in \mathcal{B}(V)$.
- Suppose T is closed. Show that TS is closed.

Definition 22. Let T be a closed operator on a Banach space V. Its *resolvent set* is

$$\rho(T) = \{\lambda \in \mathbb{C} \,:\, (\lambda - T)^{-1} : V \to \mathrm{Dom}(T) \text{ exists}\},$$

and the operator

$$R_\lambda(T) := (\lambda - T)^{-1}, \qquad \lambda \in \rho(T),$$

is called the *resolvent of T at λ*. The *spectrum* of T is

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

The *point spectrum* $\sigma_{\mathrm{p}}(T)$ is the set of eigenvalues, while the *approximate point spectrum* $\sigma_{\mathrm{ap}}(T)$ is the set of approximate eigenvalues,

$$\sigma_{\mathrm{ap}}(T) = \{\lambda \in \mathbb{C} \,:\, \exists \psi_n \in \mathrm{Dom}(T), ||\psi_n|| = 1, \text{ such that } (\lambda - T)\psi_n \to 0\}.$$

Remark. Since T is closed, its resolvents $R_\lambda(T)$ have closed graphs, and are defined on all of V. So they are actually bounded operators on V, due to the *closed graph theorem*[14].

Remark. Following the proof of Theorem 6.4, one deduces that $\rho(T)$ is open, and thus $\sigma(T)$ is a closed subset of $\mathbb{C}$.

Remark. In the concrete setting of differential operators, T has an initial domain of smooth functions, and is only a closable operator. The closure $\overline{T}$ is not always easy to describe explicitly.

Example 7.1. Let us try to define a "momentum operator"

$$T = -i\frac{d}{dx} : \mathrm{Dom}(T) \subset L^2[0,1] \to L^2[0,1].$$

The following are two possible domains, comprising absolutely continuous functions (see Section 7.1.1),

$$\begin{aligned} \mathrm{Dom}(T_1) &= \{\psi \in \mathrm{AC}[0,1] \,:\, \psi' \in L^2[0,1]\}, \\ \mathrm{Dom}(T_2) &= \{\psi \in \mathrm{Dom}(T_1) \,:\, \psi(0) = 0\}, \end{aligned}$$

with the understanding that ψ' means the almost-everywhere derivative. It may be shown that T_1, T_2 are closed operators. Their spectra are

$$\sigma(T_1) = \mathbb{C}, \qquad \sigma(T_2) = \emptyset.$$

The first claim follows from the simple observation that for each $\lambda \in \mathbb{C}$, the function $x \mapsto e^{i\lambda x}$ lies in $\mathrm{Dom}(T_1)$, and is clearly an eigenfunction of T_1. For the second claim, we leave it as an exercise to check that the resolvent $R_\lambda(T_2)$ at $\lambda \in \mathbb{C}$ is given by the operator

$$(R_\lambda(T_2)\varphi)(x) = -i\int_0^x e^{i\lambda(x-s)}\varphi(s)\,ds, \qquad \varphi \in L^2[0,1].$$

[14] If a Banach space operator $V \to V$ has closed graph, then it is continuous. The closed graph theorem is equivalent to the bounded inverse theorem.

Example 7.1 shows that $\mathrm{Dom}(T_1)$ is too large for T_1 to be a reasonable momentum observable — there are eigenfunctions with imaginary eigenvalues. So we must impose some boundary conditions to cut down the domain. The "Dirichlet" condition at $x = 0$ is unsuitable, as can be seen by the trivial spectrum of T_2. A "balanced" boundary condition is required to achieve a real spectrum, and we shall see that there is no unique choice — different choices describe different physical situations!

7.1.1 Note on absolutely continuous functions

Details can be found, e.g., in Section 7.5 of Ziemer's Modern Real Analysis book.

A function $\psi : [a, b] \to \mathbb{C}$ is *absolutely continuous* if, given any $\epsilon > 0$, there exists $\delta > 0$ such that for any finite collection of disjoint intervals $[x_i, x_i'] \subset [a, b]$ with total length less than δ, one has $\sum_i |\psi(x_i') - \psi(x_i)| < \epsilon$. Absolutely continuous functions are, in particular, uniformly continuous. We write $\mathrm{AC}[a, b]$ for the set of absolutely continuous functions on $[a, b]$.

The Fundamental Theorem of (Lebesgue integral) Calculus is:

Theorem 7.1. *Let $\psi \in \mathrm{AC}[a, b]$. Then ψ is almost everywhere differentiable, and its almost-everywhere defined derivative ψ' satisfies*

$$\psi' \in L^1[a, b], \qquad \psi(x) - \psi(a) = \int_a^x \psi'(\tilde{x})\, d\tilde{x}.$$

Conversely, let $f \in L^1[a, b]$. Then $F(x) := \int_a^x f(\tilde{x})\, d\tilde{x}$ is absolutely continuous, with $F' = f$ almost everywhere.

Suppose $\psi, \varphi \in \mathrm{AC}[a, b]$, then $\psi\varphi \in \mathrm{AC}[a, b]$ as well, and $(\psi\varphi)' = \psi'\varphi + \psi\varphi'$ almost everywhere, and is L^1. Integrating over $[a, b]$ gives the integration-by-parts rule,

$$\int_a^b \psi\varphi' = \psi(b)\varphi(b) - \psi(a)\varphi(a) - \int_a^b \psi'\varphi.$$

7.2 Adjoints of unbounded operators on Hilbert space

The adjoint operation generalizes to densely-defined operators on Hilbert spaces.

Definition 23. Let $T : \mathrm{Dom}(T) \to \mathcal{H}$ be a densely-defined operator. Define $\mathrm{Dom}(T^*)$ to be the set of $\varphi \in \mathcal{H}$ such that there exists (a unique[15]) $\eta_\varphi \in \mathcal{H}$

[15] The dense domain is needed here, to ensure uniqueness.

obeying the condition

$$\langle \eta_\varphi | \psi \rangle = \langle \varphi | T\psi \rangle, \qquad \forall \psi \in \mathrm{Dom}(T).$$

The *adjoint* operator $T^* : \mathrm{Dom}(T^*) \to \mathcal{H}$ is then defined as

$$T^*\varphi := \eta_\varphi, \qquad \varphi \in \mathrm{Dom}(T^*).$$

By design, the adjoint's domain is the maximal one for which the formal adjointness condition is true,

$$\langle T^*\varphi | \psi \rangle = \langle \varphi | T\psi \rangle, \qquad \forall \psi \in \mathrm{Dom}(T),\ \varphi \in \mathrm{Dom}(T^*). \tag{7.1}$$

Notice that the larger $\mathrm{Dom}(T)$ is, the smaller $\mathrm{Dom}(T^*)$ will become,

$$T \subset T' \ \Rightarrow\ T'^* \subset T^*. \tag{7.2}$$

The basic relation (6.4) continues to hold in the unbounded setting:

Lemma 7.2. *Let T be a densely-defined operator on a Hilbert space $\mathcal{H}$. Then* $\mathrm{Ran}(T)^\perp = \ker(T^*)$

Proof. For any $\varphi \in \ker(T^*)$,

$$\langle \varphi | T\psi \rangle = \langle T^*\varphi | \psi \rangle = 0, \qquad \forall\, \psi \in \mathrm{Dom}(T). \tag{7.3}$$

Thus $\ker(T^*) \subset \mathrm{Ran}(T)^\perp$. Conversely, suppose $\varphi \in \mathrm{Ran}(T)^\perp$, i.e.,

$$\langle \varphi | T\psi \rangle = 0 = \langle 0 | \psi \rangle, \qquad \forall\, \psi \in \mathrm{Dom}(T).$$

By definition, $\varphi \in \mathrm{Dom}(T^*)$ with $T^*\varphi = 0$. Thus $\mathrm{Ran}(T)^\perp \subset \ker(T^*)$. □

The double adjoint, T^{**}, does not make sense unless $\mathrm{Dom}(T^*)$ is dense. The latter condition turns out to be the condition of being closable.

Proposition 7.3. *Let $T : \mathrm{Dom}(T) \to \mathcal{H}$ be a densely-defined operator. Then*

1. *T^* is a closed operator.*
2. *T is closable iff $\mathrm{Dom}(T^*)$ is dense. In that case, $\overline{T} = T^{**}$.*

Proof. Define the unitary operator

$$\begin{aligned} U : \mathcal{H} \oplus \mathcal{H} &\to \mathcal{H} \oplus \mathcal{H} \\ (\psi, \tilde{\psi}) &\mapsto (-\tilde{\psi}, \psi). \end{aligned}$$

Now

$$\begin{aligned} (\varphi, \eta) \in \Gamma_{T^*} &\Leftrightarrow \langle \varphi | T\psi \rangle = \langle \eta | \psi \rangle && \forall \psi \in \mathrm{Dom}(T) \\ &\Leftrightarrow \big\langle (\varphi, \eta) \big| (-T\psi, \psi) \big\rangle_{\mathcal{H}\oplus\mathcal{H}} = 0 && \forall \psi \in \mathrm{Dom}(T) \\ &\Leftrightarrow (\varphi, \eta) \in \big(U(\Gamma_T) \big)^{\perp} \end{aligned}$$

So

$$\Gamma_{T^*} = \big(U(\Gamma_T) \big)^{\perp} \tag{7.4}$$

is an orthogonal complement, thus closed. This proves (1).

For (2), we first observe the general identity

$$\begin{aligned} \overline{\Gamma_T} = \Gamma_T^{\perp\perp} = -\Gamma_T^{\perp\perp} &= U^2(\Gamma_T^{\perp\perp}) \\ &= \big(U(U(\Gamma_T)^{\perp}) \big)^{\perp} && (U \text{ preserves } \perp) \\ &= \big(U(\Gamma_{T^*}) \big)^{\perp}. \end{aligned} \tag{7.5}$$

($\Leftarrow$) If $\mathrm{Dom}(T^*)$ is dense, T^{**} exists, and we can use (7.4) to continue (7.5),

$$\overline{\Gamma_T} = \big(U(\Gamma_{T^*}) \big)^{\perp} \overset{(7.4)}{=} \Gamma_{T^{**}}.$$

So the closure $\overline{T}$ exists and is precisely T^{**}.
($\Rightarrow$) if $\mathrm{Dom}(T^*)$ is not dense, pick a nonzero $\psi \in \mathrm{Dom}(T^*)^{\perp}$. For this ψ,

$$\begin{aligned} \big\langle (\psi, 0) \big| (\varphi, T^*\varphi) \big\rangle = 0 \quad \forall \varphi \in \mathrm{Dom}(T^*) &\Leftrightarrow (\psi, 0) \in \Gamma_{T^*}^{\perp} \\ &\Leftrightarrow (0, \psi) \in U(\Gamma_{T^*})^{\perp} \\ &\Leftrightarrow (0, \psi) \in \overline{\Gamma_T} && \text{(Eq. (7.5))}. \end{aligned}$$

Now, $(0, \psi)$ cannot lie in the graph of any linear operator. So $\overline{\Gamma_T}$ cannot be the graph of any operator, i.e., T is not closable. □

Another tricky issue is the adjoint of a product.

Exercise 7.2. Let S, T, as well as ST be densely-defined operators on a Hilbert space. Show that

$$T^* S^* \subset (ST)^*.$$

7.3 Formal self-adjointness and supersymmetry

Definition 24. A densely-defined operator H is *symmetric*, or *formally self-adjoint*, if $H \subset H^*$; explicitly,

$$\langle H\varphi|\psi\rangle = \langle\varphi|H\psi\rangle, \qquad \forall\varphi,\psi \in \mathrm{Dom}(H). \tag{7.6}$$

Exercise 7.3. Show that the eigenvalues of a symmetric operator are real, and that the eigenspaces for distinct eigenvalues are orthogonal to each other.

7.3.1 Formal harmonic oscillator

The *Schwartz space* $\mathscr{S}(\mathbb{R}^d)$ is the space of smooth functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta(x)| < \infty,$$

for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$, where

$$x^\alpha := x_1^{\alpha_1} \ldots x_d^{\alpha_d}$$
$$D^\beta f := \partial_1^{\beta_1} \ldots \partial_d^{\beta_d}.$$

A standard result in function theory, stated without proof, is that $\mathscr{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \le p < \infty$.

Definition 25. The *formal position operator* X and *formal momentum operator* P are defined on the domain $\mathscr{S}(\mathbb{R})$, by the formulae

$$(X\psi)(x) = x\psi(x), \qquad x \in \mathbb{R},$$
$$P\psi = -i\frac{d}{dx}\psi.$$

The *formal creation/annihilation operators* are, respectively, the operators

$$A^\dagger := X - iP, \qquad A = X + iP.$$

It is a simple matter to check that X and P are symmetric operators, which map $\mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$. So $A, A^\dagger, A^\dagger A, AA^\dagger$ are all defined on $\mathscr{S}(\mathbb{R})$. Although the formal adjointness relation

$$\langle A^\dagger\varphi|\psi\rangle = \langle\varphi|A\psi\rangle, \qquad \varphi,\psi \in \mathscr{S}(\mathbb{R})$$

holds between A and $A^\dagger$, the true adjoint A^* has a larger domain than $A^\dagger$.

Note that A is not a normal operator (even formally),

$$A^\dagger A = (X - iP)(X + iP) = X^2 + P^2 + i[X, P] = X^2 + P^2 - 1,$$
$$AA^\dagger = (X + iP)(X - iP) = X^2 + P^2 + i[P, X] = X^2 + P^2 + 1.$$

Here, we used the basic formal CCR, $[X, P] = i\hbar \mathbf{1}_{\mathscr{S}(\mathbb{R})}$.

Formally, the *quantum harmonic oscillator Hamiltonian* for a particle with mass $m > 0$ and frequency $\omega > 0$ is the operator

$$H_{\text{SHO}} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 X^2 = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2,$$

symmetric on the domain $\mathscr{S}(\mathbb{R})$. To simplify matters, we shall set $\hbar, m, \omega$ to 1, so we just have

$$2H_{\text{SHO}} = (P^2 + X^2) = A^\dagger A + 1 = AA^\dagger - 1.$$

7.3.2 Supersymmetric derivation of eigenvalues

A key observation is

$$2H_{\text{SHO}} - 1 = A^\dagger A, \qquad 2H_{\text{SHO}} + 1 = AA^\dagger. \tag{7.7}$$

In general, if two Hamiltonians H_1, H_2 satisfy the relation $H_1 = A^\dagger A$ and $H_2 = AA^\dagger$ for some operator A, as in (7.7), they are said to be *supersymmetric partners*[16]. The reason is the following linear algebraic observation.

Exercise 7.4. Let $A, A^\dagger : \text{Dom} \to \text{Dom}$ be two densely-defined operators which are formally adjoint to each other,

$$\langle A^\dagger \varphi | \psi \rangle = \langle \varphi | A\psi \rangle, \qquad \varphi, \psi \in \text{Dom}.$$

Show that

- $A^\dagger A$ and $AA^\dagger$ are symmetric on Dom.
- The eigenvalues of $A^\dagger A$ and $AA^\dagger$ are non-negative.

[16] The "symmetry" in "supersymmetry" is not related to "symmetry" in the sense of formally self-adjoint operators.

- $A^\dagger A$ and $AA^\dagger$ have the same *non-zero* eigenvalues, with the same algebraic multiplicities.

Thus, (7.7) says that $2H_{\mathrm{SHO}}-1$ and $2H_{\mathrm{SHO}}+1$ are supersymmetric partners whose *positive* eigenvalues coincide. Furthermore, they are just shifts of each other, so their point spectra satisfy

$$\sigma_{\mathrm{p}}(A^\dagger A)\setminus\{0\} = \sigma_{\mathrm{p}}(\underbrace{AA^\dagger}_{=A^\dagger A+2})\setminus\{0\} = \sigma_{\mathrm{p}}(A^\dagger A)+2.$$

A little thought shows that there are only two possibilities:

$$\sigma_{\mathrm{p}}(A^\dagger A) = \begin{cases}\emptyset, \\ 2\mathbb{N}_{\geq 0}.\end{cases}$$

The second possibility occurs iff $A^\dagger A$ has a 0-eigenvalue iff A has non-trivial kernel. (Note that $AA^\dagger$, thus $A^\dagger$, has no kernel.) So let us solve

$$0 = (A\psi)(x) = ((X+iP)\psi)(x) = x\psi(x) + \frac{d\psi}{dx}(x)$$
$$\Rightarrow \psi(x) \propto \psi_0(x) := e^{-x^2/2}.$$

The Gaussian ψ_0 is clearly in the Schwartz class, and we conclude that

$$\ker(A) = \mathrm{span}\{\psi_0\},$$

and therefore

$$\sigma_{\mathrm{p}}(H_{\mathrm{SHO}}) = \frac{1}{2}\sigma_{\mathrm{p}}(A^\dagger A+1) = \mathbb{N}_{\geq 0}+\frac{1}{2}.$$

The eigenfunction ψ_0 has various names in physics: *vacuum state*, *harmonic oscillator ground state*, *supersymmetry-breaking ground state*, *zero mode*, etc.

Exercise 7.5. Show that

- $\ker(A^\dagger) = \{0\}$.
- For $n \geq 1$, show that $\psi_n \in \mathscr{S}(\mathbb{R})$ is an eigenvector of $AA^\dagger$ with eigenvalue n iff $A^\dagger\psi_n \in \mathscr{S}(\mathbb{R})$ is an eigenvector of $A^\dagger A$ with eigenvalue n.
- Deduce that for $n \geq 0$, the $(n+\frac{1}{2})$-eigenspace of H_{SHO} is spanned by $(A^\dagger)^n\psi_0$, where ψ_0 is the Gaussian $\psi_0(x) = e^{-x^2/2}$.

As a preview of Dirac operators, let us rewrite the above supersymmetry argument in 2×2 matrix form,

$$D_{\mathrm{SHO}} := \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}, \qquad D_{\mathrm{SHO}}^2 = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} = \begin{pmatrix} 2H_{\mathrm{SHO}} - 1 & 0 \\ 0 & 2H_{\mathrm{SHO}} + 1 \end{pmatrix}.$$

Exercise 7.6. Let $A, A^\dagger : \mathrm{Dom} \to \mathrm{Dom}$ be two densely-defined operators which are formally adjoint to each other.

1. Show that $D := \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}$ is symmetric on $\mathrm{Dom} \oplus \mathrm{Dom}$.
2. Show that
$$\ker(D^2) = \ker(D) = \ker(A) \oplus \ker(A^\dagger).$$
3. If $\binom{\psi_1}{\psi_2}$ is an eigenvector of D with non-zero eigenvalue λ, show that $\psi_1 \neq 0 \neq \psi_2$.
4. Show that the point spectrum $\sigma_{\mathrm{p}}(D)$ is symmetric about 0, i.e.,
$$\sigma_{\mathrm{p}}(D) = -\sigma_{\mathrm{p}}(D).$$

Exercise 7.6 shows that the *nonzero* eigenvalues of D_{SHO} must occur in $\pm\lambda$ pairs. Furthermore, the eigenfunctions $\binom{\psi_1}{\psi_2}$ are nonzero in both components. These $\pm\lambda$ eigenfunctions span the $\lambda^2 > 0$ eigenspaces of $A^\dagger A$ and $AA^\dagger$ combined. However, the kernel

$$\ker(D_{\mathrm{SHO}}) = \ker(A) \oplus \ker(A^\dagger) = \ker(A) = \ker(A^\dagger A)$$

is unpaired, and occurs only in the first component of $\mathcal{S}(\mathbb{R}) \oplus \mathcal{S}(\mathbb{R})$.

The conceptual achievement of the above construction is as follows. By taking *two* copies of the second-order Schrödinger operator $2H_{\mathrm{SHO}}$, and adjusting them by appropriate lower-order terms ∓ 1, we have found a first-order "square root",

$$D_{\mathrm{SHO}} = \begin{pmatrix} 0 & X - \frac{d}{dx} \\ X + \frac{d}{dx} & 0 \end{pmatrix} = \text{“}\sqrt{}\text{”} \begin{pmatrix} 2H_{\mathrm{SHO}} - 1 & 0 \\ 0 & 2H_{\mathrm{SHO}} + 1 \end{pmatrix}.$$

It is important that this "square root" is a genuine differential operator. It is not the positive-definite square root which we will learn about later. The indefiniteness of the "square root" is the origin of *antiparticle* (negative energy) solutions to the relativistic Dirac equation (to be discussed if there is time). The first-order part, $\begin{pmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix}$ is called the *Dirac operator* on the line $\mathbb{R}$, and it acts on *spinors* rather than scalar functions.

As we have seen, once we turn on the $\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}$ "potential" term, D_{SHO} explicitly acquires an imbalance in its kernel — a zero mode occurs in the first component, but not in the second component. In general, the *supersymmetric index* of $D = \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}$ is defined as the integer

$$\mathrm{Index}(D) := \dim\ker(A) - \dim\ker(A^\dagger). \tag{7.8}$$

We had just computed $\mathrm{Index}(D_{\mathrm{SHO}}) = 1$ — the ground state space of the harmonic oscillator is the supersymmetric index space of a Dirac-type operator D_{SHO}.

The reason why the index concept is of great importance in modern physics, is its *stability*. The idea is that if the supersymmetry structure is preserved, then we can only create/annihilate nonzero eigenvalues in $\pm$ pairs, by removing/adding a null space of A and a null space of $A^\dagger$ simultaneously. So the supersymmetric index is invariant against such perturbations!

Exercise 7.7. For $m \neq 0, \beta > 0$, consider the "domain-wall Dirac operators"

$$(D_{m,\beta}\psi)(x) = \begin{pmatrix} 0 & m\tanh(\beta x) - \frac{d}{dx} \\ m\tanh(\beta x) + \frac{d}{dx} & 0 \end{pmatrix}\psi(x), \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathscr{S}(\mathbb{R}) \oplus \mathscr{S}(\mathbb{R}).$$

Determine $\ker D_{m,\beta}$, and discuss what happens as $m \to 0$ and/or $\beta \to \infty$.

Caveat: When D is a Dirac operator on a compact manifold, one may argue that its spectrum only comprises finite-multiplicity eigenvalues (explained in a later lecture), so that $\mathrm{Index}(D)$ makes sense, and the above stability argument can be made precise. In the harmonic oscillator example on the noncompact manifold $\mathbb{R}$, we showed that D_{SHO} has finite-multiplicity eigenvalues.

But we did not show *completeness* — the eigenspaces of D_{SHO} or H_{SHO} may not span all of the Hilbert space $L^2(\mathbb{R})$. Note that we cannot avoid this problem by cutting the Hilbert space down to $\mathscr{S}(\mathbb{R})$ — the latter is not a Hilbert space! The true spectrum of H_{SHO} may be bigger than the set of eigenvalues, and the naïve perturbation theory of eigenvalues may not apply.

Exercise 7.8. Let H be a symmetric operator. Suppose H^* has eigenvalue i. Show that the i-eigenspace of H^* is orthogonal to all the eigenspaces of H.

The completeness of H_{SHO} eigenspaces cannot be deduced by formal algebraic manipulation of creation/annihilation operators. It turns out to be true for H_{SHO}, and the usual way to show this is to study the explicit eigenfunctions in detail. From Exercise 7.5, we know that these eigenfunctions are of the form $p_n(x) \times e^{-x^2/2}$ for some polynomials, called *Hermite polynomials* (after a suitable normalization convention). With some Fourier analysis, one verifies that the span of $p_n(x) \times e^{-x^2/x}$ is indeed dense in $L^2(\mathbb{R})$. In general, to understand *why* the H_{SHO} eigenspaces provide a complete eigenbasis, we will need to learn more about self-adjointness, and show that H_{SHO} is "essentially self-adjoint".

In elementary treatments of quantum mechanics, one often refers to a symmetric operator as a *Hermitian* operator. A common (wrong!) claim is that a Hermitian operator has a complete set of eigenvectors spanning the Hilbert space (i.e. an orthonormal eigenbasis). One could try to rectify the above claim to something like "a Hermitian operator can be orthogonally diagonalized if we understand 'eigenvalues' in a suitably generalized sense". This is still false in the setting of unbounded operators! The spectral theorem, in the sense which is needed in quantum mechanics, only applies to (unbounded) *self-adjoint* operators, not merely symmetric operators.